MATHEMATICS MAGAZINE

CONTENTS

CODEN: MAMGA8

VOLUME 46 • JANUARY 1973 • NUMBER 1

MATHEMATICS MAGAZINE

GERHARD N. WOLLAN, *EDITOR*

EDITORIAL CORRESPONDENCE should be sent to the EDITOR, G. N. WOLLAN, Department of Mathematics, Purdue University, Lafayette, Indiana 47907. Articles should be typewritten and triple-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, A. B. WILLCOX, Mathematical Association of America, Suite 310, 1225 Connecticut Avenue, N. W., Washington, D.C. 20036.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Washington, D. C., bi-monthly except July–August. Ordinary subscriptions are: 1 year \$7.00. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of \$5.00. Single issues of the current volume may be purchased for \$1.40. Back issues may be purchased, when in print, for \$1.50.

Second class postage paid at Washington, D.C. and additional mailing offices.

MATHEMATICS MAGAZINE

Published by

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

A JOURNAL OF COLLEGIATE MATHEMATICS

VOLUME 46

1973

Published at

MENASHA, WIS. AND WASHINGTON, D. C.

COMPLEMENTS TO AN INVERSION FORMULA

D. V. WIDDER, Harvard University

1. In the last chapter of [**1**] we discussed the inversion of the integral transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{t\phi(t)}{x^2+t^2}dt. \tag{1.1}$$

We illustrate the formula by an example. A simple special case of (1.1) is provided by $f(x) = e^{-x}$, $\phi(t) = \sin t$:

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{t \sin t}{x^2+t^2}dt. \tag{1.2}$$

See formula (15), p. 65, of [**2**], for example. The inversion procedure consists of expanding $f(x)$ in power series, cancelling the even powers, and changing alternate signs in the resulting series, whose sum is then $\phi(x)$. Thus, for (1.2) we have:

$$\textit{Step 1:}\quad f(x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$\textit{Step 2:}\quad - x - \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots$$

$$\textit{Step 3:}\quad \phi(x) = \sin x = + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

More precisely, if

$$f(x) = \sum_{n=0}^{\infty} a_n x^{n-\alpha},$$

then

$$\phi(x) = \sum_{n=0}^{\infty} a_n\left[\sin\frac{\pi}{2}(\alpha-n)\right] x^n. \tag{1.3}$$

In this note let us inquire what change in the procedure is needed if the kernel $t/(x^2+t^2)$ of (1.1) is replaced by $x/(x^2+t^2)$:

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{x}{x^2+t^2}\phi(t)\,dt. \tag{1.4}$$

Of course this is really no essential change, for it is easy to see that if $[\phi(t), f(x)]$ form a transform pair for (1.1), then $[t\phi(t), xf(x)]$ do so for (1.4). Thus $\phi(t) = t \sin t$, $f(x) = xe^{-x}$ satisfy equation (1.4). What must the multipliers $\sin[(\pi/2)(\alpha-n)]$ of equation (1.3) be replaced by in the new situation?

2. To answer the question in the context of the general theory of which it is a part let us employ some elementary operational calculus. First note that the two transforms are special cases of

$$f(x) = \int_0^{\infty} K\left(\frac{x}{t}\right) \frac{\phi(t)}{t} dt. \tag{2.1}$$

This equation reduces to (1.2) if $K(x) = (2/\pi)/(x^2+1)$ and to (1.4) if $K(x) = (2/\pi)x/(x^2+1)$. It is equivalent to

$$f(e^{-x}) = \int_{-\infty}^{\infty} K(e^{-x+t})\, \phi(e^{-t})\, dt,$$

or to the general convolution transform

$$F(x) = \int_{-\infty}^{\infty} G(x-t)\,\Phi(t)dt = \int_{-\infty}^{\infty} G(t)\,\Phi(x-t)dt \tag{2.2}$$

if we set

$$F(x) = f(e^{-x}),\ G(x) = K(e^{-x}),\ \Phi(t) = \phi(e^{-t}).$$

Now denote by the letter D the operation of differentiation with respect to x. Define e^{aD} as a translation through distance a:

$$e^{aD}\phi(x) = \phi(x+a). \tag{2.3}$$

This is reasonable in the light of Maclaurin's expansion

$$e^{aD} = \sum_{n=0}^{\infty} \frac{a^n D^n}{n!}$$

$$e^{aD}\,\Phi(x) = \sum_{n=0}^{\infty} \frac{a^n}{n!}\Phi^{(n)}(x) = \Phi(x+a).$$

We treat only the special cases of (2.2) in which G has a bilateral Laplace transform, and we denote the reciprocal thereof by $E(D)$,

$$\frac{1}{E(D)} = \int_{-\infty}^{\infty} e^{-tD} G(t)dt.$$

By (2.3) e^{-tD} should represent a translation through distance $-t$ and we may define the operator $1/E(D)$ as

$$\frac{1}{E(D)}\Phi(x) = \int_{-\infty}^{\infty} e^{-tD}\, G(t)dt\, \Phi(x) = \int_{-\infty}^{\infty} \Phi(x-t)\, G(t)dt = F(x). \tag{2.4}$$

Treating D as a number, in the fashion of operational calculus, (2.4) becomes

$$\Phi(x) = E(D)\, F(x), \tag{2.5}$$

and we have inverted (2.2). This assumes that we have a suitable interpretation for the operator $E(D)$. It has been shown [3] for a large class of kernels G, which includes those of this note, that interpretation via an infinite product expansion is effective.

Let us compute $E(D)$, first for the kernel $K(x) = (2/\pi)/(x^2+1)$:

$$\frac{1}{E(D)} = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{e^{-tD}}{1+e^{-2t}}dt = \frac{1}{\pi}\int_0^{\infty}\frac{y^{D/2-1}}{1+y}dy, \qquad y = e^{-2t} \tag{2.6}$$

$$E(D) = \frac{\pi}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)} = \sin\frac{\pi}{2}D.$$

Here we have used formulas 497 and 798 of [4], for example. For the kernel $K(x) = (2/\pi)x/(x^2+1)$ we have

$$\frac{1}{E(D)} = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{e^{-tD}e^{-t}}{1+e^{-2t}}dt.$$

This is the integral (2.6) with D replaced by $(D+1)$ so that $E(D)$ here becomes $\sin[(\pi/2)\ (D+1)] = \cos\pi D/2$, and (2.5) is

$$\left(\cos\frac{\pi D}{2}\right)\int_{-\infty}^{\infty}K(e^{-x+t})\phi(e^{-t})dt = \phi(e^{-x}). \tag{2.7}$$

Since $Df(e^{-x}) = -e^{-x}f'/(e^{-x})$ we can obtain the same result by introducing a new operator $\theta = -xD$ and at the end replacing x by e^{-x}:

$$\theta f(x) = -xDf(x) = -xf'(x)\big|_{x\sim e^{-x}} = -e^{-x}f'(e^{-x}).$$

Powers of θ will stand for iteration of the operator:

$$\theta^n = \theta[\theta^{n-1}] \qquad n = 2,3,4,\ldots$$

We can thus be rid of the exponential functions in (2.7), and that equation becomes

$$\left(\cos\frac{\pi}{2}\ \theta\right)\frac{2}{\pi}\int_0^{\infty}\frac{x}{x^2+t^2}\phi(t)dt = \phi(x).$$

3. As noted above we realize the operator $\cos\pi\theta/2$ by use of the infinite product expansion

$$\cos\frac{\pi\theta}{2} = \prod_{k=1}^{\infty}\left(1-\frac{\theta^2}{(2k-1)^2}\right). \tag{3.1}$$

See, for example, formula 896 of [4]. Noting that

$$\theta x^\alpha = -x(\alpha x^{\alpha-1}) = -\alpha x^\alpha$$

$$(\theta+a)x^\alpha = (-\alpha+a)x^\alpha,$$

or more generally for any polynomial $P(\theta)$ that

$$P(\theta)x^\alpha = P(-\alpha)x^\alpha,$$

we see by use of the limit implied in (3.1) that

$$\cos\frac{\pi\theta}{2}x^{\alpha} = \left(\cos\frac{\pi\alpha}{2}\right)x^{\alpha}.$$

Hence if $f(x)$, as defined by (1.4), has the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n x^{n-\alpha} \tag{3.2}$$

we would expect that

$$\phi(x) = \cos\frac{\pi\theta}{2}f(x) = \sum_{n=0}^{\infty} a_n\left[\cos\frac{\pi}{2}(\alpha - n)\right]x^{n-\alpha}. \tag{3.3}$$

Before discussing the validity of this result let us illustrate it by an example.

By partial fractions or by the theory of residues it is easy to show that

$$\frac{x}{1+x} = \frac{2}{\pi}\int_0^{\infty}\frac{x}{x^2+t^2}\,\frac{t^2}{t^2+1}dt,$$

so that $f(x) = x/(1+x)$, $\phi(t) = t^2/(t^2+1)$ provide a transform pair for (1.4). But

$$\frac{x}{1+x} = x - x^2 + x^3 - x^4 + \cdots \tag{3.4}$$

The exponent α of (3.2) is here zero and the multipliers of (3.3) are $\cos\pi n/2$: $1, 0, -1, 0, 1, \cdots$, so that (3.4) is transformed into

$$x^2 - x^4 + x^6 - \cdots = \frac{x^2}{x^2+1} = \phi(x),$$

as predicted.

4. Rather than justify equation (3.3) as was done for the corresponding one in [**1**] we show directly the relation between the operators $\cos\pi\theta/2$ and $\sin\pi\theta/2$. This computation will be only a special case of general theory (Lemma 10.2b, p. 80 of [3]), but the reader may find the calculation of interest as applied to the familiar trigonometric functions.

As noted above, if $[\phi(x), f(x)]$ are a transform pair for (1.1) then $[x\phi(x), xf(x)]$ form a pair for (1.4). By [3]

$$\left(\sin\frac{\pi\theta}{2}\right)f(x) = \phi(x),$$

and from the above operational considerations we expect

$$\left(\cos\frac{\pi\theta}{2}\right)xf(x) = x\phi(x).$$

Eliminating $\phi(x)$ we would have

$$x\left(\sin\frac{\pi\theta}{2}\right)f(x) = \left(\cos\frac{\pi\theta}{2}\right)xf(x), \tag{4.1}$$

at least when $f(x)$ is defined by equation (1.1). We show now that this result is true for more or less *arbitrary* functions.

THEOREM 1. *Equation* (4.1) *is true when either side exists.*

From (3.1) we have

$$\cos\frac{\pi\theta}{2} = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{\theta}{2k-1}\right)\left(1+\frac{\theta}{2k-1}\right). \tag{4.2}$$

Let us note first the action of the operator $(1 + a\theta)$ when applied to the product $xf(x)$:

$$\begin{aligned}(1+a\theta)\,xf(x) &= xf - ax[xf]' = xf - axf - ax^2f' \\ &= x[(1-a)f + a\theta f] = (1-a)x\left[1+\frac{a}{1-a}\theta\right]f(x), \qquad a \neq 1,\end{aligned} \tag{4.3}$$

$$(1+\theta)xf(x) = xf - xf - x^2f' = x\theta f(x). \tag{4.4}$$

For example

$$\left(1+\frac{1}{5}\theta\right)xf = \frac{4}{5}x\left(1+\frac{\theta}{4}\right)f.$$

In computing (4.2) operators like (4.3) and (4.4) will be applied successively. After each stage a factor x will appear so that (4.3) will be applicable to the next stage, replacing f by the other factor just obtained. Thus

$$(1+\theta)\,xf = x\theta f$$

$$(1-\theta^2)\,xf = (1-\theta)\,x\theta f = 2x\left(1-\frac{\theta}{2}\right)\theta f.$$

If we abbreviate this process by simply indicating the *new* operational factor introduced at each stage, we have

$$\begin{aligned} 1+\theta &\qquad \sim \theta \\ 1-\theta &\qquad \sim 2\left(1-\frac{\theta}{2}\right) \\ 1+\frac{\theta}{3} &\qquad \sim \frac{2}{3}\left(1+\frac{\theta}{2}\right) \\ 1-\frac{\theta}{3} &\qquad \sim \frac{4}{3}\left(1-\frac{\theta}{4}\right) \end{aligned}$$

....

$$1+\frac{\theta}{2n-1} \sim \frac{2n-2}{2n-1}\left(1+\frac{\theta}{2n-2}\right)$$

$$1-\frac{\theta}{2n-1} \sim \frac{2n}{2n-1}\left(1-\frac{\theta}{2n}\right)$$

$$\prod_{k=1}^{n}\left(1-\frac{\theta}{2k-1}\right)\left(1+\frac{\theta}{2k-1}\right)$$

$$\sim \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\cdots\frac{2n-2}{2n-1}\,\frac{2n}{2n-1}\,\theta\left(1+\frac{\theta}{2n}\right)\prod_{k=1}^{n-1}\left(1-\frac{\theta}{2k}\right)\left(1+\frac{\theta}{2k}\right).$$

The numerical factor on the right is the familiar product of Wallis which tends to $\pi/2$ as n becomes infinite, p. 376 of [**5**], so that the right hand side tends to

$$\frac{\pi}{2}\theta\prod_{k=1}^{\infty}\left(1-\frac{\theta^2}{(2k)^2}\right)=\sin\frac{\pi\theta}{2},$$

formula 895 of [**4**]. This completes the proof of the theorem and thus validates the conjecture (3.3) through the proofs in [**1**].

5. The transform (1.4) bears the same relation to the cosine transform as (1.1) does to the sine transform, p. 232 of [**1**]. We state the result without proof and illustrate it.

THEOREM 2. *If $\phi(t)$ is continuous and absolutely integrable on $(0,\infty)$ and if*

(5.1) $$f(x)=\frac{2}{\pi}\int_0^\infty \phi(t)\cos xt\,dt,$$

then

(5.2) $$\cos\frac{\pi\theta}{2}\int_0^\infty e^{-xt}f(t)dt=\phi(x).$$

The classical inversion of (5.1),

$$\phi(t)=\int_0^\infty f(x)\cos xt\,dx,$$

p. 49 of [**6**], is not applicable under present hypotheses since it requires some local condition on $\phi(t)$ like bounded variation.

As an example of Theorem 2 consider

$$xe^{-x}=\frac{2}{\pi}\int_0^\infty\frac{1-t^2}{(1+t^2)^2}\cos xt\,dt.$$

One way to obtain this is to use formula (11), p. 8, of [**2**],

$$e^{-ax}=\frac{2}{\pi}\int_0^\infty\frac{a}{a^2+t^2}\cos xt\,dt,$$

and differentiate with respect to a. Equation (5.2) becomes

$$\cos\frac{\pi\theta}{2}\int_0^\infty e^{-xt}te^{-t}dt = \cos\frac{\pi\theta}{2}\,\frac{1}{(x+1)^2}$$

$$= \cos\frac{\pi\theta}{2}\,[1-2x+3x^2-4x^3+\cdots].$$

With the cosine multipliers 1, 0, -1, 0, $\cdots$ this series becomes

$$1-3x^2+5x^4-7x^6+\ldots=\frac{1-x^2}{(1+x^2)^2}=\phi(x).$$

For this simple example the classical inversion *is* applicable:

$$\frac{1-x^2}{(1+x^2)^2}=\int_0^\infty t\,e^{-t}\cos xt\,dt.$$

See formula (5), p.14 of [**2**].

References

1. D. V. Widder, An Introduction to Transform Theory, Academic Press, New York, 1971.
2. A. Erdélyi, Table of Integral Transforms, vol. 1, McGraw-Hill, New York, 1954.
3. I. I. Hirschman and D. V. Widder, The Convolution Transform, Princeton University Press, Princeton, New Jersey, 1955.
4. B. O. Pierce, A Short Table of Integrals, Ginn, Boston, 1956.
5. D. V. Widder, Advanced Calculus, Prentice-Hall, Englewood Cliffs, New Jersey, 1961.
6. S. Bochner, Fourier integrals, Ann. of Math., (1949).

THE HEPTAGONAL TRIANGLE

LEON BANKOFF, Los Angeles, California and
JACK GARFUNKEL, Forest Hills High School, N. Y.

Three vertices of a regular heptagon can be connected to produce four distinct species of triangles, three of them isosceles and the fourth scalene. The latter, which we shall call "the Heptagonal Triangle", is uniquely characterized by vertices whose angles, $A=\pi/7$, $B=2\pi/7$, $C=4\pi/7$, belong to a geometric progression with a common ratio of 2. A survey of the properties of this triangle will provide a better understanding of the regular polygon from which it is derived.

Since the earliest days of recorded mathematics, the regular heptagon has been virtually relegated to limbo. One could easily conjure up a variety of plausible reasons for this neglect. For example, unlike the equilateral triangle, the square and the regular pentagon, the regular heptagon cannot serve as a face of a regular polyhedron and as a result has had rather limited exposure to public view. Unlike the regular pentagon and the regular decagon, its properties, though striking, do not approach those associated with the Divine Proportion, a ratio second only to π in mathematical significance. Furthermore, unlike the equilateral triangle, the square and the regular hexagon, it has not enjoyed centuries of ornamental utility as a tiling for plane surfaces. To militate further against the acceptance of the heptagon into the community of familiar polygons, mathematicians have shown an aversion toward the study of the seven-sided figure perhaps because of their understandable inability to construct the figure with the well-known conventional Euclidean tools. But nonconstructibility is not necessarily synonymous with nonexistence, although many a frustrated scholar may have suspected this to be the case after a fruitless quest for heptagon material in the literature of mathematics.

The history of research on the regular heptagon from ancient times until the end of the nineteenth century could easily be encapsulated in one short paragraph. According to Arabian sources, Archimedes is believed to have written a book on the heptagon inscribed in a circle. If it is true that this work ever existed, it now seems to be irretrievably lost. Still, the question of its having been written appears credible because of a single surviving proposition, namely a "neusis" or "verging" construction of a regular heptagon. Archimedes accomplished this brilliant feat by using a marked instead of an unmarked ruler and by placing a certain line segment of definite length at a specially manipulated position in relation to certain other points and lines. Details elucidating this vague description may be found in Heath's *Manual of Greek Mathematics* on pages 340–2 of the Dover reprint. The same source describes Heron's approximate construction in which the apothem of an inscribed regular hexagon is considered to be almost equal to the side of a regular heptagon inscribed in the same circle. The apothem is equal to approximately 0.866026 times the side of the hexagon and would have to be stretched to only 0.867726 in order to qualify in a practical way as the side of the heptagon. Except for a downright silly and fallacious construction published by Thomas Hobbes, the eminent English philosopher, in his book entitled *A Garden of Geometrical Roses*, printed in London in 1727, the literature on heptagons is utterly barren. In 1796 the researches of the 19-year old Gauss inadvertently lent a hand in consigning the heptagon to oblivion simply because 7 happens to be a prime number that cannot be expressed in the form $2^{2^n}+1$. In other words, the study of the regular heptagon was further discouraged by the belated proof of its nonconstructibility with ruler and compasses.

In 1913 the late Victor Thébault of Tennie, France, directed his attention to an investigation of the long dormant heptagon and succeeded in bringing to light many surprising properties of great esthetic interest. He was attracted to this venture by the example set by Morley's theorem, a beautiful proposition that arrived rather late on the geometrical scene probably because of the unconscious taboo associated with the forbidden angle trisection. (Morley's theorem states that the intersections of the adjacent internal or external trisectors of a triangle are vertices of an equilateral triangle.) The purpose of this paper is to assemble a number of Thébault's more interesting discoveries and to make available in English the essence of material hitherto published only in French. A further purpose is to offer some original theorems and to review several heptagon problems, particularly those involving theorems previously published without proof in various editorial notes.

We start with a problem that exhibits a startling connection between the regular heptagon and the square inscribed in the same circle. (See the AMERICAN MATHEMATICAL MONTHLY, problem E 1154, 1955, 584). While it shows how the side of an inscribed square can be precisely derived from elements of an inscribed regular heptagon, it offers no encouragement to wishful mystics who may still be yearning for the feasibility of the reverse procedure. The proposal offered by Victor Thébault was as follows:

The distance from the midpoint of side AB of a regular convex heptagon ABCDEFG inscribed in a circle to the midpoint of the radius perpendicular to BC and cutting this side, is equal to half the side of a square inscribed in the circle. (Figure 1.)

Two solutions were published, one invoking the cosine law and the other using complex numbers. In the first solution, let d denote the required distance, R the circumradius and $\theta = \pi/7$. By the cosine law, we have

$$d^2 = R^2(1/4 + \cos^2\theta - \cos\theta\cos 2\theta).$$

Since $\sin 3\theta = \sin 4\theta$, it follows that

$$3\sin\theta - 4\sin^3\theta = 4\sin\theta\cos\theta\cos 2\theta,$$

which reduces to

$$(*) \qquad \cos^2\theta - \cos\theta\cos 2\theta = 1/4.$$

Hence $d = R\sqrt{2}/2$.

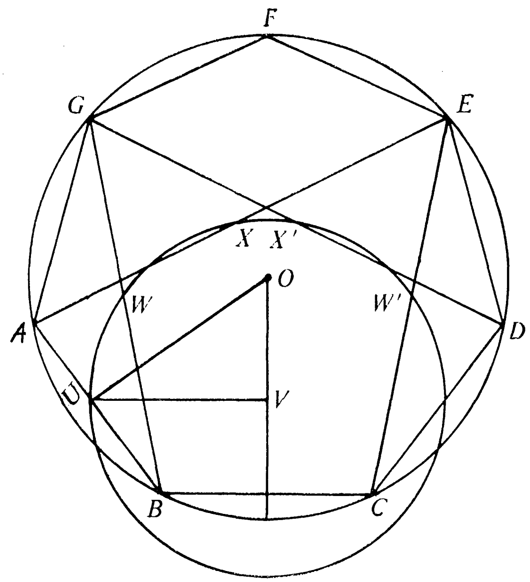


FIG. 1.

(*Note.* The relation indicated by (*) applies uniquely to the regular heptagon and will be used later in this paper in the development of other heptagon properties. Indeed, it holds if θ is replaced by $n\theta$, where n is any integer or zero.)

The second published solution, offered by Hüseyin Demir, designates the vertices $F, G, A, \cdots, E$ as the affixes of the 7th roots $1, e, e^2, \cdots, e^6$ of unity. Then the midpoints U, V, of AB and the concerned radius correspond to $u = (e^2 + e^3)/2$ and $v = -1/2$, whence

$$UV^2 = (u - v)(\bar{u} - \bar{v}) = (1 + e^2 + e^3)(1 + e^4 + e^5)/4$$

$$= (2 + 1 + e + \cdots + e^6)/4 = 1/2,$$

thus establishing the proposition.

Extending this method to diagonals, Demir states the following: *The midpoints of the sides of the hexagon ABGDCEA are equidistant from the point V, the common distance being half the side of the inscribed square.* This means that the circle of radius UV, centered at V, bisects the segments AB, BG, GD, DC, CE, and EA. Since the midpoints W, X of BG and AE are symmetrical to those of CE and DG with respect to the diameter through V, the proof of this corollary may be abbreviated by showing that $VW = VX = UV$. The method of the first solution is also applicable to this proof.

Thébault mentioned the following additional properties of the regular heptagon $ABCDEFG$. Let O be the center of the heptagon, W the midpoint of OF, M the point diametrically opposite F, U the midpoint of AB, V the midpoint of OM, and J the point on UB produced such that $UJ = UM$ (Figure 2). Then:

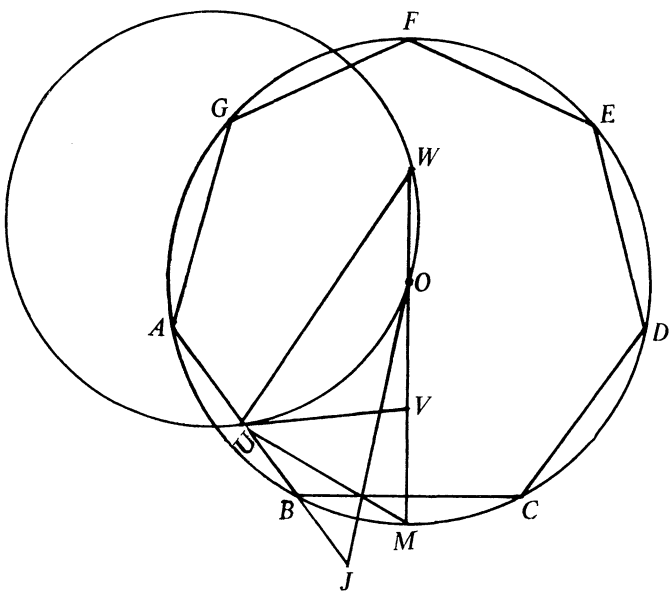


FIG 2.

(1) UW is equal to the diagonal of the square constructed on an apothem of the heptagon as a side.

(2) OJ is equal to the diagonal of the square constructed on half the side of the inscribed equilateral triangle.

(3) UV is tangent to the circle through U, O, W.

The proofs of (1) and (2) are easily obtained by the use of the identity (*). The relation $VO \cdot VW = UV^2$ proves (3).

The impact created by these astonishing relationships is evidenced by the editorial comment: "One cannot help but wonder if these properties, and the property of the problem, are just remarkable geometric accidents, or whether they are special cases of more general theorems involving, perhaps, other regular polygons. Karst did some work on the regular inscribed nonagon $ABCDEFGHI$. Here, if U is the midpoint of side AB and V is the midpoint of the radius perpendicular to and cutting side BC, it can be shown that angle $OUV = 30°$."

In response to an inquiry about the origin of these unusual discoveries, a letter from Thébault dated January 3, 1956 offered the following explanation and incidentally revealed two additional relationships resembling the one in problem E 1154:

"Let ABC be the triangle in which $A = \pi/7$, $B = 2\pi/7$, and $C = 4\pi/7$. Let M be the point of intersection of the internal bisector of A with the circumcircle $(O)R$; let N be the point of intersection of $(O)R$ with the internal bisector of B. The triangle MCN is curious—also the quadrangle $OMCN$. For the latter—the squares of the bimedians (lines joining the midpoints of the opposite sides) have as expressions: $R^2/2$ (E 1154), $MN^2/2$, $MC^2/2$. That is the origin of E 1154···" (Figure 3).

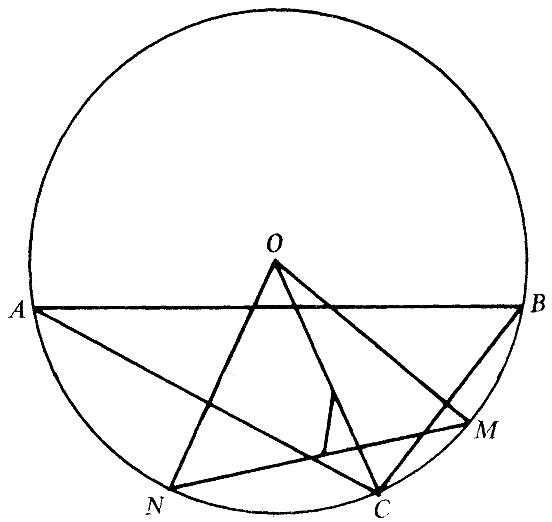


Fig. 3.

To assist those seeking to verify the relationships mentioned by Thébault, we note that one of the bimedians connects the midpoints of the diagonals MN

and CO. The proofs make generous use of the formula (*) in addition to a judicious manipulation of the sine and cosine sum-to-product identities.

Additional properties of the heptagonal triangle.

1. *The sum of the squares of the sides of the heptagonal triangle is equal to* $7R^2$, *where* R *is the circumradius of the triangle.*

Applying the sine law to the sides a, b, c opposite the angles $A = \pi/7$, $B = 2\pi/7$, $C = 4\pi/7$, and converting the resulting sine ratios to the corresponding cosines, we obtain

$$(**) \qquad \cos A = b/2a \qquad \cos B = c/2b \qquad \cos C = -a/2c,$$

so that $\cos A \cos B \cos C = -1/8$. Then

$$\begin{aligned} a^2 + b^2 + c^2 &= 4R^2(\sin^2 A + \sin^2 B + \sin^2 C) \\ &= 4R^2(2 + 2\cos A \cos B \cos C) \\ &= 4R^2(7/4) = 7R^2. \end{aligned}$$

Other methods of solution may be found in the February 1957 issue of the AMERICAN MATHEMATICAL MONTHLY, pp. 110–112, problem E 1222.

2. *If* A', B', C' *denote the feet of the altitudes from* A, B, C, *the orthic triangle* $A'B'C'$ *is similar to triangle* ABC *and each side of the former is half the length of the corresponding side of the latter.* (Figure 4.)

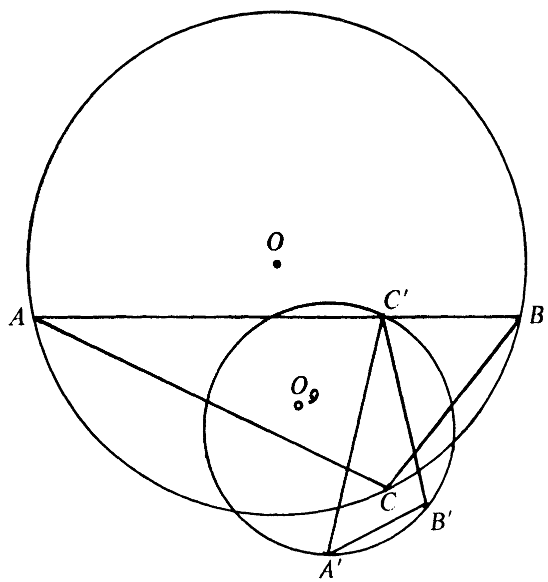


FIG. 4.

In the cyclic quadrilateral $CB'BC'$, angle $A'C'B'/2$ = angle $CBB' = 90° - A - B = \pi/14 = A/2$. Similarly, angle $A'B'C' = C$ and angle $B'A'C' = B$. Thus the triangles are similar. Observe that the circumcircle of triangle $A'B'C'$ is the nine-point circle of triangle ABC. Since the radius of the nine-point circle is half that of the circumcircle, all the linear elements of the orthic triangle are half those of the corresponding elements of the parent triangle.

The heptagonal triangle is the only obtuse triangle displaying orthic similarity. As for acute triangles, the equilateral triangle is the only one similar to its orthic triangle. Here again, strangely enough, the ratio of similitude is 2. A problem relating to orthic similarity is number 681, on page 219 of the September 1968 issue of this MAGAZINE.

3. *If a, b, c are the sides of the heptagonal triangle ABC in which $C = 2B = 4A$, the side a is half the harmonic mean of the other two sides.*

Since $A = \pi/7$, it follows that $\sin 3A = \sin 4A$. Then

$$\sin A = \frac{\sin 2A}{2\cos A} = \frac{\sin 2A \sin 4A}{2\cos A \sin 3A} = \frac{\sin 2A \sin 4A}{\sin 2A + \sin 4A}.$$

With $a/\sin A = b/\sin B = c/\sin C = 2R$, we obtain $a = bc/(b+c)$, the required result. Equivalent expressions for b and c are $b = ac/(c-a)$ and $c = ab/(b-a)$.

See problem 189 in the Spring 1968 issue of the *Pi Mu Epsilon Journal* for two other treatments of the solution.

4. *If h_a, h_b, h_c are the altitudes to the sides a, b, c of the heptagonal triangle ABC, then $h_a = h_b + h_c$.*

Expressing the results of the preceding property in the form $1/a = 1/b + 1/c$ and using the relations $h_a = 2S/a$, $h_b = 2S/b$, $h_c = 2S/c$, where S is the area of triangle ABC, we obtain the result $h_a = h_b + h_c$.

5. The following list of fundamental properties of the heptagonal triangle will be useful in deriving others. In each case, $A = \pi/7$, $B = 2\pi/7$ and $C = 4\pi/7$.

$$\sin A \sin B \sin C = \sqrt{7}/8.$$

$$\sin^2 A + \sin^2 B + \sin^2 C = 7/4.$$

$$\sin 2A + \sin 2B + \sin 2C = \sqrt{7}/2.$$

$$\sin^2 A \sin^2 B \sin^2 C = 7/64$$

$$\sin^2 A \sin^2 B + \sin^2 A \sin^2 C + \sin^2 B \sin^2 C = 7/8$$

$$\cos A \cos B \cos C = -1/8.$$

$$\cos^2 A + \cos^2 B + \cos^2 C = 5/4$$

$$\cos^2 A \cos^2 B + \cos^2 A \cos^2 C + \cos^2 B \cos^2 C = 3/8.$$

$$\cos 2A + \cos 2B + \cos 2C = -1/2$$

$$\sin A + \sin B + \sin C = \sqrt{(7/2)}$$

$$\tan A \tan B \tan C = -\sqrt{7}.$$

$$\cot A + \cot B + \cot C = \sqrt{7}.$$

$$\csc^2 A + \csc^2 B + \csc^2 C = 8.$$

$$\sec^2 A + \sec^2 B + \sec^2 C = 24.$$

$$\cot^2 A + \cot^2 B + \cot^2 C = 5.$$

$$\tan^2 A + \tan^2 B + \tan^2 C = 21.$$

$$\sec^4 A + \sec^4 B + \sec^4 C = 416.$$

$$\cos^4 A + \cos^4 B + \cos^4 C = 13/16.$$

$$\sin^4 A + \sin^4 B + \sin^4 C = 21/16.$$

$$\csc^4 A + \csc^4 B + \csc^4 C = 32.$$

$$\sec 2A + \sec 2B + \sec 2C = -4.$$

To obtain the above relations various methods are available. For example, one can start with the expansion of $\sin 7x$ in terms of powers of $\sin x$ to obtain the equation

$$64x^7 - 112x^5 + 56x^3 - 7x = 0,$$

the roots of which are 0, $\pm \sin \pi/7$, $\pm \sin 2\pi/7$, $\pm \sin 4\pi/7$. Then the fractions 7/4, 7/8, 7/64 represent the sums of $\sin^2 A$, $\sin^2 B$ and $\sin^2 C$ taken one, two and three at a time.

Similar procedures combined with the use of well-known standard trigonometric identities lead to expressions involving the other trigonometric functions.

6. *The sum of the squares of the altitudes of the heptagonal triangle is equal to half the sum of the squares of the sides of the triangle.*

To prove that $h_a{}^2 + h_b{}^2 + h_c{}^2 = (a^2 + b^2 + c^2)/2$ convert the altitudes and the sides to their trigonometric equivalents in terms of sines and substitute the numerical values already obtained for $\Sigma \sin^2 A \sin^2 B$ and for $\Sigma \sin^2 A$.

7. *The cotangent of the Brocard angle V of the heptagonal triangle is equal to* $\sqrt{7}$.

Here again we make use of previously derived results. If S denotes the area and V the Brocard angle of the heptagonal triangle ABC, $\cot V = \cot A + \cot B + \cot C = (a^2 + b^2 + c^2)/4S$. The terms of this identity correspond to their numerical equivalents given in paragraph 5.

8. *If a, b, c are the sides of the heptagonal triangle ABC, then $b^2 - a^2 = ac$, $c^2 - b^2 = ab$, and $c^2 - a^2 = bc$.*

Applying the relation $a = c\cos B + b\cos C$ to the values of $\cos B$ and $\cos C$ used in (**) of paragraph 1, we obtain $a = (c^2 + ab - b^2)/2b$, which reduces to $ab = c^2 - b^2$. Similarly, $b^2 - a^2 = ac$ and $c^2 - a^2 = bc$.

Combining these results and the relation $\sin^2 x = 1 - \cos^2 x$with the values of the cosines in terms of the sides given in property (**), we obtain $\sin^2 A = (3a-c)/4a$, $\sin^2 B = (3b-a)/4b$ and $\sin^2 C = (3c-b)/4c$.

9. *In the heptagonal triangle ABC whose sides are a, b, c, we have*

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5.$$

This follows immediately from (**), where $b/a = 2\cos A$, $c/b = 2\cos B$ and $a/c = -2\cos C$.

10. *If a, b, c are the sides of the heptagonal triangle, then*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{2}{R^2}.$$

This is a direct consequence of the relation $\csc^2 A + \csc^2 B + \csc^2 C = 8$, a property listed in paragraph 5.

11. *If A', B', C' are the feet of the altitudes issuing from the vertices A, B, C of the heptagonal triangle, then* $BA' \cdot A'C = ac/4$, $CB' \cdot B'A = ab/4$, *and* $AC' \cdot C'B = bc/4$.

This property follows from (**) of paragraph 1.

12. *The exradius r_a relative to the vertex A of the heptagonal triangle ABC is equal to the radius of the nine–point circle of triangle ABC.* (Figure 5).

Let M, N, P denote the contacts of the excircle $(I_a)r_a$ with the sides AC, CB, AB respectively. The sides of the triangle MNP are parallel to those of the orthic triangle $A'B'C'$ because their corresponding sides are perpendicular to the same angle bisectors of the heptagonal triangle ABC. Hence the triangles ABC, $A'B'C'$ and MNP are similar.

The line NP meets AC in Q and the triangles MAP, MQP, QAP and MQN are isosceles. It follows that

$$AP = AM = AQ + QM = QP + QM = QN + NP + QM = MN + NP + PM$$

and the triangles $A'B'C'$, MNP, having the same perimeter, are equal; their circumcircles have the same radius.

The Euler relation then gives

$$OI_a{}^2 = R^2 + 2Rr_a = 2R^2,$$

R being the radius of the circumcircle of triangle ABC. Thus I_a is situated on the orthoptic circle of the circle (O), that is, the circle concentric with (O) and with a radius equal to $R\sqrt{2}$.

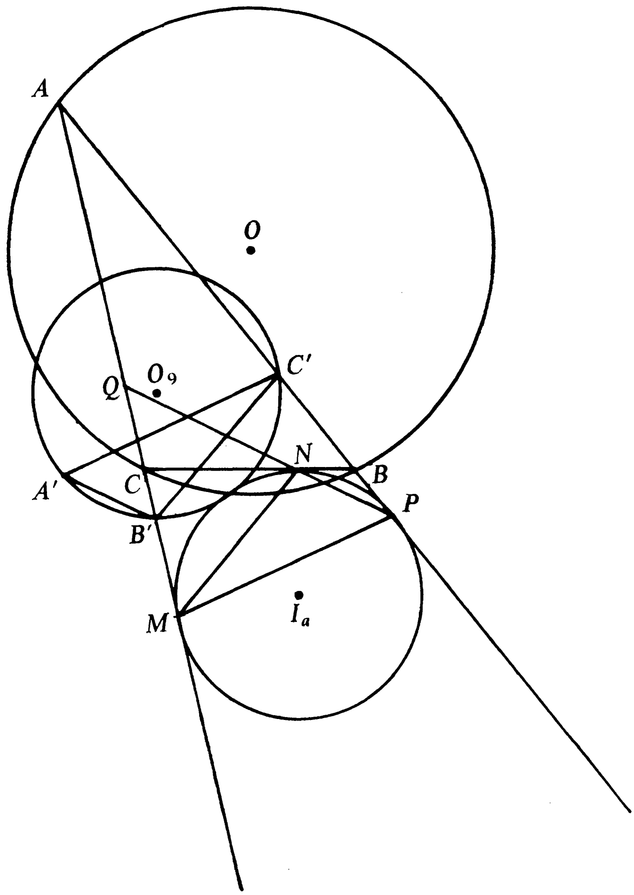


FIG. 5.

The equality of the triangles $A'B'C'$ and MNP permits us to say that the lines NC', PB' and MA' are tangent to the nine-point circle and that the quadrilaterals $MA'C'P$, $MB'C'N$ and $PB'A'N$ are parallelograms.

13. *The internal angle bisectors of the angles C and B are equal respectively to the difference of the two adjacent sides; the external angle bisector of A is equal to the sum of the adjacent sides.*

From B draw a perpendicular to the internal bisector of angle A, cutting BC in K; the triangles KAB, KCB and KFB are isosceles, where F is the foot of the internal bisector of angle B. We then find that

$$AB = AK = AF + FK = BF + KB = BF + BC,$$

whence $BF = AB - BC$.

Similarly, we have $CG = CA - BC$, where CG is the internal bisector of angle C; also, $AL = CA + AB$, AL being the external bisector of angle A.

14. *The triangle formed by joining the feet of the internal angle bisectors of the heptagonal triangle ABC is isosceles.* (Figure 6.)

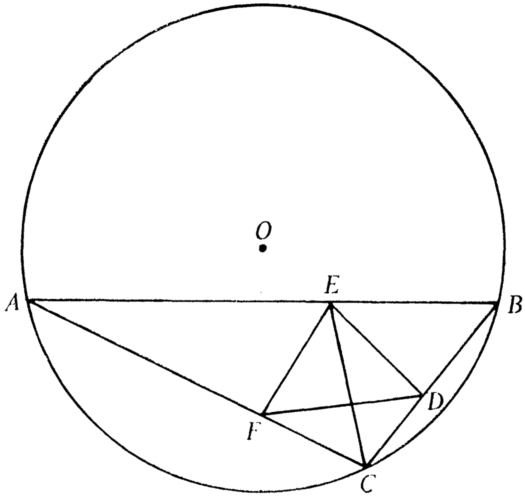


FIG. 6.

Let D, E, F denote the feet of the interval angle bisectors issuing from A, B, C. We combine the relations $c = ab/(b-a)$ and $ab = c^2 - b^2$ of sections 3 and 8 respectively to yield $b/(a+c) = c/(b+c)$. Now $EC = ab/(a+c)$ and $BD = ac/(b+c)$. Hence $EC = BD$. Also, $FB = FC$, since the base angles FBC and FCB are each equal to $2\pi/7$. With the equality of angles FCE and FBC, the triangles FCE and FBD are congruent and $FE = FD$.

15. *The orthic triangle $A'B'C'$ and the medial triangle $M_1M_2M_3$ are congruent and in perspective.* (Figure 7.)

The orthic similarity of the heptagonal triangle (with the ratio of similitude equal to 2) establishes the congruency of triangles $A'B'C'$ and $M_1M_2M_3$. Since the vertices of the two triangles are six points of the nine-point circle, the parallelism of the lines $C'M_1$, M_3B' and M_2A' is easily established by noting the equality of the angles $C'M_1B$, $M_2A'B$ and $(M_3B', A'B)$. An interesting sidelight is the inverse similarity of the triangles $A'C'M_1$ and $A'B'C'$.

16. *The triangle II_bI_c formed by the incenter of the heptagonal triangle and the excenters relative to B and C is similar to the triangle ABC, to its orthic triangle and to the pedal triangle of the nine-point center of triangle ABC.* (Figure 8.)

The proof follows easily from the comparison of angles in the cyclic quadrilaterals I_bAIC and I_cBIA.

17. The properties in this section are stated without proof. Most of the derivations are not difficult.

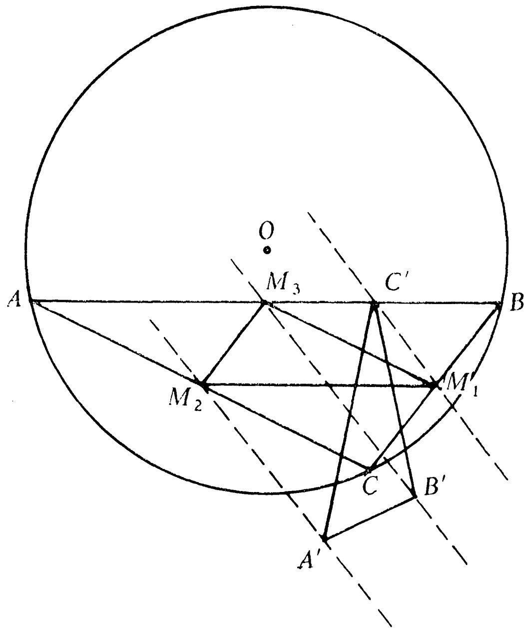


FIG. 7.

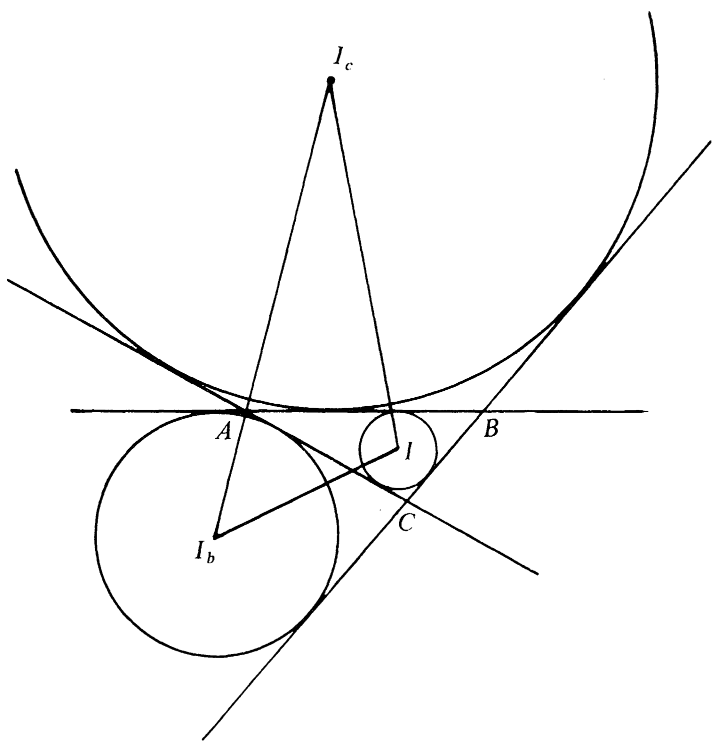


FIG. 8.

(a) The first Brocard point of the heptagonal triangle is the center of the nine-point circle and the second Brocard point lies on this circle.

(b) The segment of the Euler line contained between the circumcenter and the orthocenter is equal to the diagonal of the square constructed on the radius of the circumcircle. Stated differently, $OH = R\sqrt{2}$.

(c) The segment connecting the incenter and the orthocenter of the heptagonal triangle is measured by the relation $IH^2 = (R^2 + 4r^2)/2$.

(d) The two tangents from the orthocenter to the circumcircle of the heptagonal triangle are mutually perpendicular.

(e) The center of the circumcircle of the tangential triangle coincides with the symmetric of the point O with respect to H.

(f) The altitude from B is half the length of the internal bisector of angle A.

The properties we have considered do not by any means exhaust the curiosities associated with the heptagonal triangle. The references listed at the end of this paper should be of assistance to any reader able and willing to explore the subject further in French journals.

Following the example set by Heron in offering an approximation to the side of a regular heptagon by using the apothem of a regular hexagon, we add a few oddities of our own:

(a) If I is the incenter of the heptagonal triangle ABC, BI is a good approximation for the side of a regular enneagon inscribed in the same circle.

(b) One-third of the length of the median to side BC is a good approximation for the side of a regular hendecagon inscribed in the same circle.

(c) Denoting the centroid of the heptagonal triangle by G and the circumcenter by O, the segment OG is a good approximation for the side of a regular triskadecagon inscribed in the same circle.

We conclude with the observation that the heptagonal triangle results when the three "quadratic residue powers" ρ^1, ρ^4, ρ^2 of the primitive 7th root of unity, $\rho = e^{2\pi i/7}$, are connected by line segments in the complex plane. This suggests generalizations of the heptagonal triangle to the convex polygons spanning the rth-power-residues among the nth roots of unity, and relates the subject intimately to Gaussian Sums.

References

As mentioned in the text, expository material on the heptagonal triangle has until now been published only in French. References pertaining to problems that have appeared in American mathematical journals have been given in the body of this paper. The sources in French are found in Mathesis, 1913–204; 1938–169; 1950–344; 1955–78 and 329; 1956–106 and 149.

INFINITESIMALS AND INTEGRATION

A. H. LIGHTSTONE, Queen's University and Yale University

1. Number systems. As Leibniz and his successors knew, the concepts of calculus are easily explained, and easily grasped, if only the right number system is assumed. The *right* number system is one that involves both infinitely large numbers and infinitely small numbers. Lacking the proper mathematical tools (i.e., mathematical logic) the mathematicians of those times were unable to demonstrate the existence of this number system. With the demand for rigor of the nineteenth century, the marvellous insights of Leibniz were banished from mathematics and were replaced by the sound but abstruse constructions of Weierstrass. Of course, the engineers and physicists who use mathematics largely ignored these innovations and continued the traditions of Leibniz, thinking in terms of infinitesimals.

In 1960 Abraham Robinson, the father of nonstandard analysis, discovered that by forming an enormously large postulate-set, a simple application of mathematical logic establishes the existence of a number system of the sort used by Leibniz and his followers. The idea is to gather together all statements in the language of the real number system (we shall characterize this language later) that are true for the real number system $\mathscr{R}$, as well as statements that collectively postulate the existence of a number greater than each natural number, e.g., $\omega > 1$, $\omega > 2$, $\omega > 3$, and so on. Now, the *compactness theorem* of mathematical logic asserts that a set of statements of this sort has a model if each of its finite subsets has a model. Here, think of a *model* as a number system (i.e., a set of numbers together with various concepts, each characterized by a set, concerning that number-set), and think of a set of statements as a set of axioms for this number system.

It is easy to see that the real number system itself is a model of each finite subset S of Robinson's postulate-set. The point is that each member of S either is a statement that is true for $\mathscr{R}$, or has the form $\omega > n$ where $n \in N$. Since S is finite, it contains a finite number of statements of this form, say

$$(1) \qquad \omega > n_1,\ \omega > n_2, \cdots, \omega > n_k.$$

Let $j = \max\{n_1, n_2, \cdots, n_k\}$ and interpret ω as the natural number $j + 1$. The postulates listed under (1) become:

$$(2) \qquad j+1 > n_1,\ j+1 > n_2, \cdots, j+1 > n_k.$$

Of course, each of these statements is true for $\mathscr{R}$. This means that $\mathscr{R}$ is a model of S. Thus, by the compactness theorem, Robinson's postulate-set has a model $\mathscr{R}^*$.

The number system $\mathscr{R}^*$ is an extension of the real number system $\mathscr{R}$ such that:

(a) $\mathscr{R}^*$ contains both infinitely large and infinitely small numbers.

(b) Each statement of our language that is true for $\mathscr{R}$, is true for $\mathscr{R}^*$ when interpreted in $\mathscr{R}^*$.

The idea of *interpreting* a statement in a mathematical system is not new. For example, the statement "$\forall x[x + 0 = x]$" is true for any group $\mathscr{G}$ provided it is interpreted in $\mathscr{G}$. This means that $+$ is interpreted as the group operation of $\mathscr{G}$, 0 is interpreted as the group identity, and the quantifier $\forall$ is interpreted as referring to the set of group elements, the supporting set of $\mathscr{G}$.

Traditionally, we think of a number system as involving a number-set, operations of addition and multiplication, the *less than* relation, and possibly certain significant numbers, e.g., 0 and 1. For example, the natural number system is usually regarded as the system $(N, +, \cdot, <, 1)$ where $N = \{1, 2, 3, \cdots\}$; the number 1 is displayed because it generates all the natural numbers. The real number system is usually represented by the system $(R\ +, \cdot, <, 0, 1)$ where R is the set of all real numbers; here, 0 and 1 are displayed because of their widely acclaimed algebraic properties. Now, Robinson's idea was to include in the real number system all concepts associated with real numbers; e.g., the power set of R, the natural numbers, the power set of N, the concept of a finite tuple of real numbers, the operation of summing the terms of a finite tuple, the function concept, the concept of a continuous function. So we obtain an expanded real number system, which we denote by $\mathscr{R}$, namely:

$$(R, \mathscr{P}R, N, \mathscr{P}N, T, +, \cdot, <, S, F, C, \cdots).$$

Here T is the set of all finite tuples whose terms are real numbers, S is the operation of summing the terms of a member of T (i.e., S is the set of all ordered pairs for which the first term is a member of T, say α, and the second term is the real number obtained by summing the terms of α), F is the set of all functions, and C is the set of all continuous functions.

Since $\mathscr{R}^*$ derives its existence from Robinson's postulate-set, each of its concepts is linked to a concept of $\mathscr{R}$ (the link *is* the postulate-set). Indeed, each concept of $\mathscr{R}^*$ is an extension of the corresponding concept of $\mathscr{R}$ in a two-fold sense that we shall clarify in a moment. To be specific, we regard $\mathscr{R}^*$ as having the form:

$$(R^*, (\mathscr{P}R)^*, N^*, (\mathscr{P}N)^*, T^*, +^*, \cdot^*, <^*, S^*, F^*, C^*, \cdots).$$

Here R^* is a proper superset of R. We can prove that N^* is a proper superset of N; in this sense, N^* is an extension of N. However, a concept of $\mathscr{R}^*$ may be an extension of the corresponding concept of $\mathscr{R}$ in a second sense. For example, F^* contains functions that are supersets of the corresponding functions in F (e.g., the *squaring* function); as well, F^* contains functions that are not related in this way to any member of F (e.g., the Dirac *delta* function).

The basic idea is that each concept of a number system is characterized by a set, indeed is identified with a set. For example, the concept of addition is identified with the mapping that associates with each ordered pair of real numbers, their sum. At this level we are not concerned with how to determine the sum of two numbers; rather our approach is that there is a definite number associated with a given pair of numbers, and called their sum. This mapping, which is a set, represents the concept of addition. Similarly, the *function* concept is identified with the set of all functions.

Since Robinson's postulate-set is the key to $\mathscr{R}^*$, we now briefly outline the language of $\mathscr{R}$, the real number system. Let A be a concept of $\mathscr{R}$ and let a be an object (e.g., number, tuple, function); then "$a \in A$" is an *atomic* statement. Moreover, this statement is *true* in case the object a is a member of the set A; otherwise, "$a \in A$" is said to be *false*. For example, "$(2,3) \in <$" and "$((2,3),5) \in +$" are true atomic statements. We shall abbreviate these statements by writing "$2 < 3$" and "$2 + 3 = 5$", following the usual mathematical conventions.

Next, we utilize the atomic statements and the logical connectives:

$$\sim \text{(not)}, \ \vee \text{ (or)}, \ \wedge \text{ (and)}, \rightarrow \text{(if} \cdots \text{then)}, \leftrightarrow \text{(if and only if)},$$

$$\forall \text{ (for each)}, \ \exists \text{ (there exists)}$$

to construct additional statements of our language. Here is our definition:

(1) Each atomic statement is a statement.

(2) If p and q are statements, so are $\sim p$, $p \vee q$, $p \wedge q$, $p \rightarrow q$, and $p \leftrightarrow q$.

(3) If $P(x)$ is a statement-form, then $\forall x\, P(x)$ and $\exists x\, P(x)$ are statements.

(4) Each statement possesses only a finite number of instances of logical connectives.

Here, a *statement-form* is an expression that yields a statement whenever its placeholder is replaced by a suitable object; for example, "$3 < x$" is a statement-form and yields statements such as "$3 < 7$" which is true, and "$3 < 2$" which is false. It is easy to formulate rules that allow us to compute the truth-value of each statement defined in (2) and (3). For example, $\sim p$ is true in case p is false, and is false in case p is true; $\forall x\, P(x)$ is true just in case each statement $P(a)$ obtained from the statement-form $P(x)$ is true.

The connectives $\forall$ and $\exists$ are called *quantifiers*; $\forall$ is the *universal* quantifier and $\exists$ is the *existential* quantifier. We must realize that the objects that may be substituted for a placeholder in a statement-form, come from a set—the set over which we quantify; so each instance of each quantifier refers to a set of objects. Usually, each quantifier refers to the number-set of our number system. However, we greatly enhance the scope of our language by setting up several sets of this sort, e.g., the real numbers, the positive real numbers, the natural numbers, the set of all subsets of R, the set of all subsets of N, the set of all functions. Such sets are called *supporting* sets of our number system. We can quantify over each supporting set as we wish, provided that we indicate the supporting set that a particular quantifier refers to. This can be done typographically.

Notice that our language approximates the language of elementary mathematics. We illustrate this with some examples:

$2 < 3$

$\forall xy[x + y = y + x]$ (addition is commutative)

$\forall xyz[x < y \wedge y < z \rightarrow x < z]$ ($<$ is transitive)

$\forall\varepsilon\exists\delta\forall xy[|x-y|<\delta\rightarrow|f(x)-f(y)|<\varepsilon]$ (f is uniformly continuous)

$\forall\varepsilon\exists n_0\forall mn[m,n>n_0\rightarrow|a_m-a_n|<\varepsilon]$ ((a_n) is a Cauchy sequence)

$\forall A[1\in A\wedge\forall x[x\in A\rightarrow x+1\in A]\rightarrow A=N]$ (Principle of Mathematical Induction)

Here Greek letters denote positive real numbers, lower-case Latin letters near the end of the alphabet indicate real numbers, lower-case Latin letters near the middle of the alphabet denote natural numbers, and upper-case Latin letters indicate subsets of N.

We shall now present some basic information about $\mathscr{R}^*$; notice our use of (b) as a fundamental method of proving facts about $\mathscr{R}^*$. By definition, $\mathscr{R}^*$ contains an infinite number ω which is greater than each natural number; since each real number is less than some natural number, we see that ω is greater than each real number. More generally, we say that a number ∞ is *infinite* if $|\infty|>h$ for every $h\in R$. By an *infinitesimal* we mean any number ε such that $|\varepsilon|<h$ for every positive real number h. We say that $a\simeq b$ (read "a approximates b") if $a-b$ is an infinitesimal. Notice that this relation allows us to abbreviate the statement "ε is an infinitesimal" by writing "$\varepsilon\simeq 0$". It is easy to prove that $\simeq$ is an equivalence relation on R^*. We say that a number is *finite* if it is not infinite. Finite numbers are important because of the following fact.

FUNDAMENTAL THEOREM ABOUT FINITE NUMBERS. *Each finite number is approximated by a unique real number.*

To prove this, let t be any finite number and consider $\{y|\ y\in R \text{ and } y<t\}$. By the completeness theorem for the real number system, this set has a least upper bound a. It is easy to verify that $a\simeq t$; moreover, since $\simeq$ is an equivalence relation, it follows that no other real number approximates t.

We mention that if ∞ is an infinite natural number then so is $\infty+1$; moreover $\infty+1\neq\infty$, indeed $\infty+1>\infty$. To prove statements of this sort we appeal to the powerful proof-technique provided by our enormous postulate-set. Clearly $n<n+1$ for every $n\in N$; this is true for $\mathscr{R}$, so it is true for $\mathscr{R}^*$ when interpreted in $\mathscr{R}^*$, i.e., $n<n+1$ for every $n\in N^*$. Now $\infty\in N^*-N$, so $\infty<\infty+1$. The trichotomy law is true for $\mathscr{R}$, so it is true for $\mathscr{R}^*$ when interpreted in $\mathscr{R}^*$; thus $\infty\neq\infty+1$.

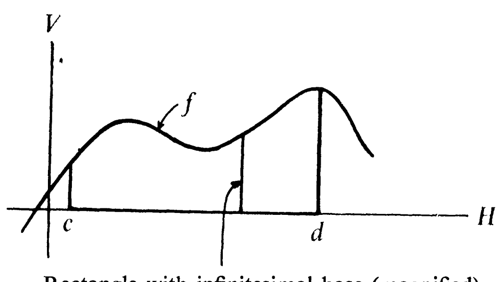
Notice that each infinite natural number has a multiplicative inverse. This is due to the fact that each nonzero real number has a multiplicative inverse. Let ∞ be any infinite natural number; $\infty\neq 0$ so ∞ has a multiplicative inverse, say ε. It is easy to prove that ε is an infinitesimal. Of course $1/(\infty+1)\neq 1/\infty$; i.e., the multiplicative inverses of ∞ and $\infty+1$ are distinct infinitesimals; this is due to the fact that distinct natural numbers have distinct inverses. Clearly, if ε is an infinitesimal so are ε^2, and ε^3, indeed ε^n whenever $n\in N$, and ε^∞ whenever ∞ is an infinite natural number. It is easy to prove that the sum of two infinitesimals is an infinitesimal; also, the product of an infinitesimal and a finite number is an infinitesimal. The product of an infinitesimal and an infinite number may be finite or infinite, depending upon the numbers involved. Just as there are infinitely many infinite numbers, so there are infinitely many infinitesimals; indeed, the multiplicative inverse of each infinite number is an infinitesimal.

We have already mentioned that the operation S of summing the terms of a finite tuple, is a concept of $\mathscr{R}$. Here "$(\alpha, t) \in S$" means that t is the sum of the terms of α. Following the usual mathematical practice, we shall express this statement by writing "$t = S_1^m a_n$" where a_n is the nth term of α, a tuple with m terms (here "n" is a placeholder). Of course, the placeholder that appears in the expression "$S_1^m a_n$" can be replaced by any other suitable symbol; we sometimes write "$S_1^m a_i$".

The corresponding concept of $\mathscr{R}^*$, denoted by S^*, possesses all properties of S that can be expressed within our language. Normally we shall suppress the *'s provided it is clear from the context that we are dealing with $\mathscr{R}^*$. Notice that the concepts S and T of $\mathscr{R}$ are linked in the sense that S sums the terms of each tuple in T. Just so, S^* and T^* are linked, i.e., S^* sums the terms of each tuple in T^*.

Here are some examples. We can sum the terms of the n-tuples $(1, \cdots, 1), (1, \cdots, n)$, $(1^2, \cdots, n^2)$, and $(1^3, \cdots, n^3)$ for each $n \in N$. Indeed $S_1^n 1 = n$, $S_1^n i = n(n+1)/2$, $S_1^n i^2 = n(n+1)(2n+1)/6$, and $S_1^n i^3 = n^2(n+1)^2/4$ for each $n \in N$. Just so, $S_1^\infty 1 = \infty$, $S_1^\infty i = \infty(\infty+1)/2$, $S_1^\infty i^2 = \infty(\infty+1)(2\infty+1)/6$, and $S_1^\infty i^3 = \infty^2(\infty+1)^2/4$ for each infinite natural number ∞. We have summed the terms of the ∞-tuples $(1, \cdots, 1)$, $(1, \cdots, \infty)$, $(1^2, \cdots, \infty^2)$, and $(1^3, \cdots, \infty^3)$. We shall need these facts in the next section.

2. Area of a region; continuity. Here we shall motivate the concept of the definite integral that we present in Section 3. Now, the concept of the definite integral of a function is a generalization of the concept of the area of the region bounded on one side by the graph of the function involved, and on its remaining sides by straight lines (see Figure 1).



Rectangle with infinitesimal base (magnified)

FIG. 1.

The basic idea is to replace the given region by a rectangular region, i.e., a region composed of rectangles, whose area we can determine in a straightforward way. Of course, we choose a rectangular region that approximates, in some sense, the given region. More precisely, we subdivide the interval $[c, d]$ (see Figure 1) into infinitely many subintervals each of infinitesimal length. We then erect a rectangle on each of these subintervals, whose height is the value of the function at some member, or endpoint, of the interval. In this way we obtain a rectangular region that approximates the given region; the area of this rectangular region is the sum of the areas of the rectangles involved. Notice that we must sum infinitely many infinitesimals; in our first example, we show how this is done.

Example 1. Compute the area of the region bounded by the graph of x^2, ordinates at 1 and 2, and H.

Solution. We shall consider two rectangular regions, one that contains the given region, the other is contained within the given region. If the given region has an area, then it is a number between the areas of these rectangular regions. First, choose an infinite natural number, say ∞. Next, divide the interval $[1,2]$ into ∞ subintervals each of length $\varepsilon = 1/\infty$, namely

$$[1, 1+\varepsilon],\ [1+\varepsilon,\ 1+2\varepsilon], \cdots, [1+(\infty-1)\varepsilon, 2].$$

Each of these closed intervals will form the base of one of our rectangles. The height of each rectangle is obtained by evaluating the given function x^2 at some point of its base. Notice that we obtain a rectangular region that contains the given region by choosing the right-hand endpoint of each of our subintervals; whereas we form a rectangular region contained within the given region by choosing the left-hand endpoint of each subinterval. Let U be the area of the first rectangular region; then

$$\begin{aligned} U &= S_1^\infty \varepsilon(1+n\varepsilon)^2 = \varepsilon S_1^\infty (1 + 2n\varepsilon + n^2\varepsilon^2) = \varepsilon S_1^\infty 1 + 2\varepsilon^2 S_1^\infty n + \varepsilon^3 S_1^\infty n^2 \\ &= \varepsilon\infty + 2\varepsilon^2 \infty(\infty+1)/2 + \varepsilon^3 \infty(\infty+1)(2\infty+1)/6 \\ &= 1 + (1+\varepsilon) + (1+\varepsilon)(2+\varepsilon)/6 = 7/3 + 3\varepsilon/2 + \varepsilon^2/6. \end{aligned}$$

Let L be the area of the second rectangular region mentioned above. Now,

$$L = S_1^\infty \varepsilon(1+[n-1]\varepsilon)^2 = U + \varepsilon - \varepsilon(1+\infty\varepsilon)^2 = U - 3\varepsilon = 7/3 - 3\varepsilon/2 + \varepsilon^2/6.$$

Our intuition suggests that the area A of the given region is a real number such that $L < A < U$; there is just one real number with this property, namely $7/3$. We conclude that $A = 7/3$. Notice here that $A \simeq L$ and $A \simeq U$.

Next, we face up to the problem of *defining* the concept of the area of a region. Motivated by our success in handling Example 1, where we assumed that the given region possesses an area, we shall focus on rectangular regions. However, we shall adopt a very severe requirement: namely, we shall insist that the area of each of our rectangular regions (which we shall spell out in a moment) approximates the same real number. This real number we take to be the area of the given region.

Our first job is to describe the rectangular regions involved. To be specific consider the region bounded by the graph of a function f, ordinates at c and d, where $c < d$, and H; we shall assume that f is nonnegative on $[c,d]$. Here, the closed interval $[c,d]^*$ plays a key role; for this interval provides us with the base of each of our rectangles. An acceptable rectangular region is constructed as follows. First, choose any infinite natural number, say ∞, and partition $[c,d]^*$ into ∞ subintervals $I_1,\cdots,I_\infty$ each of infinitesimal length. Next, we associate with each of these intervals, one of its members or one of its endpoints. This is achieved by actually pairing with an interval, the number associated with it. Let t_k be the number associated with I_k whenever $1 \leqq k \leqq \infty$; then we form the set of ordered pairs $\{(I_1, t_1),\cdots,(I_\infty,t_\infty)\}$ which we call a *rectangle-builder.* The first terms of members of this set provide us with the bases of our rectangles; the second terms lead us, via the given function f, to the height of each of our rectangles. For example, the ordered pair (I_k, t_k) yields the rectangle with base I_k and height $f(t_k)$; the area of this rectangle is $f(t_k)|I_k|$, where $|I|$ denotes the length of an interval I. In this way, each rectangle-builder $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$ yields a rectangular region whose area is $S_1^\infty f(t_n)|I_n|$.

Summarizing, the set $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$ is a *rectangle-builder over* $[c,d]$ if

(i) $\{I_1,\cdots,I_\infty\}$ is a partition of $[c,d]^*$ into ∞ subintervals, where ∞ is an infinite natural number.

(ii) $|I_n| \simeq 0$ whenever $1 \leqq n \leqq \infty$.

(iii) $t_n \in I_n$ or t_n is an endpoint of I_n whenever $1 \leqq n \leqq \infty$.

We are now ready to define the concept of the *area* of a region of the sort under discussion. Let f be nonnegative on $[c,d]$. Then L is said to be the area of the region bounded by the graph of f, ordinates at c and d, and H, provided that L is a real number and $L \simeq S_1^\infty f(t_n)\ |I_n|$ whenever $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$ is a rectangle-builder over $[c,d]$.

So, a region of this sort possesses an area provided that the area of each rectangular region yielded by a rectangle-builder over $[c,d]$ and the function involved, approximates the same real number.

We shall prove that a region of this sort has an area if the function involved is continuous over $[c,d]$. Using our extended number system $\mathscr{R}^*$ we can characterize continuity in a simple and direct way. A function, say f, is *continuous* at a real number a, where $a \in \mathscr{D}_f$, provided that:

$$f(a) \simeq f(b) \text{ whenever } a \simeq b \text{ and } b \in \mathscr{D}_{f^*}. \tag{1}$$

A function is *continuous on a set of real numbers* if it is continuous at each member of that set. So f is continuous on a set E if

$$f(a) \simeq f(b) \text{ whenever } a \simeq b,\ a \in E,\ b \in \mathscr{D}_{f^*}. \tag{2}$$

Compare this to the definition of *uniform* continuity.

A function f is uniformly continuous on a real interval I provided that:

$$f(a) \simeq f(b) \textit{ whenever } a \simeq b \textit{ and } a,b \in I^*. \tag{3}$$

The point is that our simple and direct definitions of continuity and uniform continuity allow us to establish their key properties in a meaningful and straightforward manner. For example, x^2 is continuous on R since $(a+\varepsilon)^2 = a^2 + 2a\varepsilon + \varepsilon^2 \simeq a^2$ whenever $a \in R$ and $\varepsilon \simeq 0$. If f and g are continuous at a, and if $\varepsilon \simeq 0$, then

$$[fg](a+\varepsilon)=f(a+\varepsilon)g(a+\varepsilon)\simeq f(a)g(a)=[fg](a)$$

since $f(a+\varepsilon)\simeq f(a)$ and $g(a+\varepsilon)\simeq g(a)$ by assumption; so fg is continuous at a. We can easily prove that f is uniformly continuous on a closed interval I if f is continuous on I. Let $a \simeq b$ and $a, b \in I^*$; by the Fundamental Theorem about Finite Numbers, there is a real number c such that $c \simeq a$ and $c \simeq b$, moreover $c \in I$. But f is continuous on I, so $f(c)\simeq f(a)$ and $f(c)\simeq f(b)$; thus $f(a)\simeq f(b)$. This proves that f is uniformly continuous on I.

Later, we shall need the following fact about continuous functions. Each function that is continuous on a closed interval has a maximum value and a minimum value on that interval; i.e., if f is continuous on a closed interval I, there are members of I, say s and t, such that $f(s)\geqq f(a)$ whenever $a\in I$, and $f(t)\leqq f(a)$ whenever $a \in I$. To prove this, let $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$ be a rectangle-builder over I, and consider the ∞-tuple $(f(t_1),\cdots,f(t_\infty))$. This tuple has a largest term, say its nth term $f(t_n)$ (here $n\in N^*$). This is due to the fact that each finite tuple of real numbers has a largest term (remember, we can handle infinite tuples just like finite tuples). Now, there is a real number s that approximates t_n. We claim that $f(s)\geqq f(a)$ whenever $a\in I$. Clearly, given a there is a member of N^*, say m, such that $a\in I_m$; thus $f(a) \simeq f(t_m)$ since f is continuous on I. But $f(t_m)\leqq f(t_n)\simeq f(s)$; it follows that $f(a)\leqq f(s)$ since both these numbers are real. This proves that f has a maximum value on I. Applying this result to $-f$, we see that f has a minimum value on I, provided f is continuous on I.

3. The definite integral. The problem we face is to decide of a region, of the type considered in Section 2, whether it has an area. To allow us to see the essentials of the situation clearly, we introduce the concept of the definite integral, which centers on the function that provides one boundary of the region. This way, the complications of geometry are eliminated at a single stroke, and we are free to concentrate on what really counts.

Let f be any function whose domain contains the closed interval $[c,d]$. We say that f is *integrable* over $[c,d]$ if there is a real number L such that $L\simeq S_1^\infty f(t_n)|I_n|$ whenever $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$ is a rectangle-builder over $[c,d]$. The number L, if it exists, is denoted by $\int_c^d f$. We do not insist that f be nonnegative on $[c,d]$.

Clearly, the region bounded above by the graph of f, below by H, and by ordinates at c and d, has an area if and only if f is integrable over $[c,d]$. So questions about area are reduced to questions about the integrability of a function.

To illustrate our definition of the definite integral of a function, let us show that the constant function 5 is integrable over $[0,1]$. Choose any rectangle-builder $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$; now

$$S_1^\infty 5|I_n| = 5\,S_1^\infty|I_n| = 5$$

since $S_1^\infty|I_n|$ is the length of the interval $[0,1]$. We conclude that the constant function 5 is integrable over $[0,1]$ and that $\int_0^1 5 = 5$.

On the other hand, g is not integrable over $[0,1]$ where g is the function such that

$$g(t)=\begin{cases}0 \text{ if } t \text{ is rational}\\ 1 \text{ if } t \text{ is irrational.}\end{cases}$$

Choose a rectangle-builder $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$ where each t_n is rational; here $S_1^\infty g(t_n)|I_n|=0$. Next, choose a rectangle-builder $\{(I_1,s_1),\cdots,(I_\infty,s_\infty)\}$ where each s_n is irrational; here $S_1^\infty g(s_n)|I_n|=S_1^\infty|I_n|=1$. So g is not integrable over $[0,1]$.

This example raises the following question. Are we sure that each subinterval of the rectangle-builders involved has a rational member, in the first case, or an irrational member in the second case? Since the interval with endpoints ε and 2ε, where $\varepsilon\simeq 0$, has no real members, we must somehow classify nonreal numbers *rational* or *irrational*. This is easy; moreover we can prove that each interval in fact possesses both rational and irrational members that are nonreal. We need an appropriate fact about $\mathscr{R}$. Here it is: "Between any two real numbers there is a rational number and an irrational number". Notice that we are involved with the concept of a rational number, and the concept of an irrational number—concepts of $\mathscr{R}$. By construction, $\mathscr{R}^*$ involves corresponding concepts that we label by the same name; so each member of R^* is rational or irrational (since this is true for $\mathscr{R}$), and between any two numbers there is a rational number and an irrational number (this is a statement about $\mathscr{R}^*$). We conclude that each subinterval of a rectangle-builder contains both rational numbers and irrational numbers. One more question. Can we be sure that $g(t_n)=0$ whenever t_n is a rational nonreal number? Again, we observe that the corresponding statement is true for $\mathscr{R}$, so it is true for $\mathscr{R}^*$ when interpreted in $\mathscr{R}^*$.

Our main goal is to prove that each continuous function is integrable. First, we must prove that $S_1^\infty \varepsilon_n a_n\simeq 0$ if ∞ is an infinite natural number, each $\varepsilon_n\simeq 0$, and if $S_1^\infty|a_n|$ is finite. Let h be any positive real number, and let k be a real number such that $S_1^\infty|a_n|<k$. Clearly,

$$|S_1^\infty\varepsilon_n a_n|\leqq S_1^\infty|\varepsilon_n|\;|a_n|<S_1^\infty\;\frac{h}{k}|a_n|\;=\frac{h}{k}S_1^\infty|a_n|<h$$

which proves that $S_1^\infty\varepsilon_n a_n$ is an infinitesimal.

Throughout the remainder of this discussion, f is a function that is continuous on $[c,d]$. We shall prove that f is integrable over $[c,d]$. First, we show that each of our infinite sums $S_1^\infty f(t_n)|I_n|$ is approximated by some real number. Next, we prove that varying the t's does not affect this real number. Finally, we prove that varying the partition involved also does not affect the real number that approximates our infinite sum.

For the first point, recall that f has both a maximum value M and a minimum value m on $[c,d]$; i.e., $m\leqq f(t)\leqq M$ whenever $c\leqq t\leqq d$. Thus

$$m\,S_1^\infty|I_n|\leqq S_1^\infty f(t_n)|I_n|\leqq M\,S_1^\infty|I_n|$$

so $m(d-c) \leqq S_1^\infty f(t_n)\,|I_n| \leqq M(d-c)$ whenever $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$ is a rectangle-builder over $[c,d]$. Both $m(d-c)$ and $M(d-c)$ are real numbers, so $S_1^\infty f(t_n)\,|I_n|$ is finite. Therefore, by the Fundamental Theorem about Finite Numbers, $S_1^\infty f(t_n)\,|I_n|$ is approximated by a real number.

Next, let $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$ and $\{(I_1,s_1),\cdots,(I_\infty,s_\infty)\}$ be rectangle-builders over $[c,d]$ that involve the same partition of $[c,d]$. By assumption $s_n \simeq t_n$ for each n, moreover f is uniformly continuous on $[c,d]$; so $f(s_n) \simeq f(t_n)$ for each n. Thus, corresponding to each n, there is an infinitesimal ε_n such that $f(s_n) = f(t_n) + \varepsilon_n$. So

$$S_1^\infty f(s_n)\,|I_n| = S_1^\infty (f(t_n)+\varepsilon_n)\,|I_n| = S_1^\infty f(t_n)\,|I_n| + S_1^\infty \varepsilon_n |I_n| \simeq S_1^\infty f(t_n)\,|I_n|$$

since $S_1^\infty\,|I_n| = d-c$ which is finite. Thus $S_1^\infty f(s_n)\,|I_n| \simeq S_1^\infty f(t_n)\,|I_n|$.

This result is useful because it allows us to concentrate on the partition of $[c,d]$ involved, ignoring the members of each subinterval at which we evaluate f. Now, we can represent a partition of $[c,d]$, say $\{I_1,\cdots,I_\infty\}$, by listing in increasing order the endpoints of the subintervals $I_1,\cdots,I_\infty$. For example, let a_0 and a_1 be the endpoints of I_1, let a_1 and a_2 be the endpoints of $I_2,\cdots$, let $a_{\infty-1}$ and a_∞ be the endpoints of I_∞; then the infinite tuple $(a_0,\cdots,a_\infty)$ characterizes the partition $\{I_1,\cdots,I_\infty\}$. This provides us with a convenient and useful way of referring to a partition of $[c,d]$. Now, each partition $(a_0,\cdots,a_\infty)$ leads us to a real number, the real number that approximates each of the infinite sums associated with a rectangle-builder involving this partition. We must show that the real numbers associated in this way with different partitions, say $(a_0,\cdots,a_\infty)$ and $(b_0,\cdots,b_{\infty_1})$, are the same. The idea is to intermesh the terms of the given partitions according to size, deleting duplicates; this yields $(c_0,\cdots,c_{\infty_2})$, a refinement of the given partitions. Here $c_0 = a_0 = b_0$, $c_1 = \min\{a_1,b_1\}$, and so on. Our process of intermeshing the terms of two tuples works as follows: from the tuples $(2,7,9,10,15)$ and $(2,4,5,8,9,12,13)$ we obtain $(2,4,5,7,8,9,10,12,13,15)$.

Each infinite sum formed from a rectangle-builder involving the partition $(c_0,\cdots,c_{\infty_2})$ lies between two sums formed from rectangle-builders involving the partition $(a_0,\cdots,a_\infty)$, one obtained by choosing t's so as to maximize the value of f over each subinterval, the other by choosing t's so as to minimize the value of f over each subinterval. We have already seen that these sums differ by an infinitesimal; so any sum formed from a rectangle-builder involving $(c_0,\cdots,c_{\infty_2})$ approximates each sum based on the partition $(a_0,\cdots,a_\infty)$. By the same argument, each sum formed from a rectangle-builder involving $(c_0,\cdots,c_{\infty_2})$ approximates each sum based on the partition $(b_0,\cdots,b_{\infty_1})$. Since $\simeq$ is transitive, it follows that each sum based on the partition $(a_0,\cdots,a_\infty)$ approximates each sum based on the partition $(b_0,\cdots,b_{\infty_1})$. So the same real number is associated with both partitions.

We have proved the following statement.

THEOREM. *Let f be continuous on $[c,d]$. There is a real number L such that $L \simeq S_1^\infty f(t_n)\,|I_n|$ whenever $\{(I_1,t_1),\cdots,(I_\infty,t_\infty)\}$ is a rectangle-builder over $[c,d]$.*

In other words, f is integrable over $[c,d]$ if f is continuous on $[c,d]$.

Well, we have demonstrated that a region of the sort described in Section 2 has an area if the function involved is continuous. Moreover, we have shown that if f is continuous on $[c,d]$, we can compute $\int_c^d f$ by merely computing $S_1^\infty f(t_n)\,|I_n|$ for a single rectangle-builder. Of course, $\int_c^d f$ is the real number that approximates $S_1^\infty f(t_n)\,|I_n|$.

To illustrate this last point we shall compute $\int_1^2 x^2$. Let ∞ be an infinite natural number, let $\varepsilon = 1/\infty$, and let $\{I_1,\cdots,I_\infty\}$ be the partition of $[1,2]$ such that $I_1 = [1, 1+\varepsilon)$, $I_2 = [1+\varepsilon,\ 1+2\varepsilon), \cdots, I_\infty = [1+(\infty-1)\varepsilon, 2]$; so $\{(I_1, 1+\varepsilon),\cdots, (I_\infty, 1+\infty\varepsilon)\}$ is a rectangle-builder over $[1,2]$. Then $\int_1^2 x^2 \simeq S_1^\infty \varepsilon(1+n\varepsilon)^2 \simeq 7/3$ (see Example 1), so $\int_1^2 x^2 = 7/3$.

Of course, it is important to establish the *Fundamental Theorem of Calculus*, which provides a simple method of evaluating $\int_c^d f$, i.e., computing the real number that approximates $S_1^\infty f(t_n)\,|I_n|$. This involves *integral* functions, the *derivative* of a function, indeed the concept of an *antiderivative*, the fact that the derivative of an integral function is its integrand, and the *mean value theorem*. The point is that f has an antiderivative if f is continuous on the interval involved, i.e., there is a function g such that $g' = f$; so

$$\begin{aligned} S_1^\infty f(t_n)\,|I_n| &= f(t_1)(a_1-a_0) + f(t_2)(a_2-a_1) + \cdots + f(t_\infty)(a_\infty - a_{\infty-1}) \\ &= [g(a_1)-g(a_0)] + [g(a_2)-g(a_1)] + \cdots + [g(a_\infty)-g(a_{\infty-1})] \\ &= g(a_\infty) - g(a_0) \end{aligned}$$

provided the t's have been chosen with the mean value theorem in mind, i.e., $g(a_n) - g(a_{n-1}) = g'(t_n)(a_n - a_{n-1})$, $n = 1,\cdots,\infty$. Thus

$$\int_c^d f = g(d) - g(c)$$

where g is an antiderivative of f, provided f is continuous on $[c,d]$.

Finally, we want to ensure that there is a rectangle-builder over a closed interval, say $[0,1]$. Here we need the concept of a finite partition of $[0,1]$ consisting of subintervals. Let P be the set that exemplifies this notion. Now, for each natural number n there is a member of P consisting of n subintervals each of length $1/n$. Since this is true for $\mathscr{R}$, it is true for $\mathscr{R}^*$ when interpreted in $\mathscr{R}^*$. So, for each member of N^*, say n, there is a member of P^* consisting of n subintervals each of length $1/n$. In particular, let ∞ be an infinite natural number; so, there is a member of P consisting of ∞ subintervals each of length $1/\infty$. Since we can associate with an interval one of its endpoints, we obtain a rectangle-builder over $[0,1]$.

References

1. Abraham Robinson, Introduction to Model Theory and to the Metamathematics of Algebra, North-Holland, Amsterdam, 1963.

2. ———, Nonstandard Analysis, North-Holland, Amsterdam, 1966.

THE MINIMUM PATH AND THE MINIMUM MOTION OF A MOVED LINE SEGMENT

MICHAEL GOLDBERG, Washington, D. C.

1. The problems. In *Problems in Modern Mathematics* by S. M. Ulam (Interscience Publ.) 1964, page 79, the following problem in the calculus of variations is posed:

Suppose two segments are given in the plane, each of length one. One is asked to move the first segment continuously, without changing its length, to make it coincide at the end of the motion with the second given interval in such a way that the sum of the lengths of the two paths described by the end points should be a minimum. What is the general rule for this minimum motion?

In the problem as stated, it is not clear whether retracing of parts of the path are permitted. Therefore, it breaks down into two different problems, namely, one in which the *path* is minimized, and another in which the *motion* is minimized.

2. The possible solutions. Consider the motion of the unit line segment from the position AB to the position $A'B'$ in which the distance AA' is shorter than BB', or equal to BB'. In Figure 1, the shortest path and the shortest motion consists simply of the straight line segments AA' and BB'.

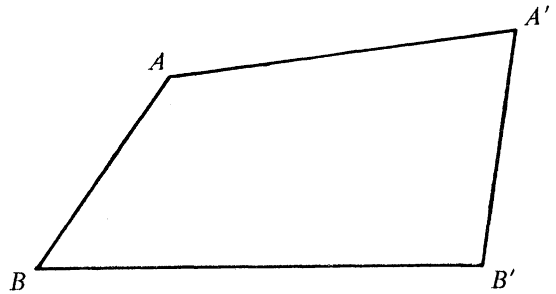


FIG. 1. No pivot point.

In Figure 2, the line segment AB is moved by parallel displacement to $A'D$. Then, the segment is moved from $A'D$ to $A'B'$ by rotation about A'. A shorter path is obtained by drawing a tangent line from B to the arc DB' and touching DB' at C. Then, retaining the line AA' as the path of A, the path BCB' is shorter than BDB'. There are, however, still shorter paths than the one shown in Figure 2.

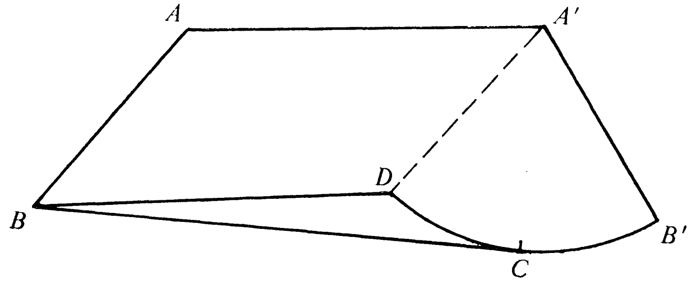


FIG. 2. One pivot point on path.

Consider the different motions shown in Figures 3 and 4. In Figure 3, the *path* length (track or road) is the same as the motion, and it is equal to APA' plus $BCC'B'$. In Figure 4, however, although the total length of the path is the same as in Figure 3,

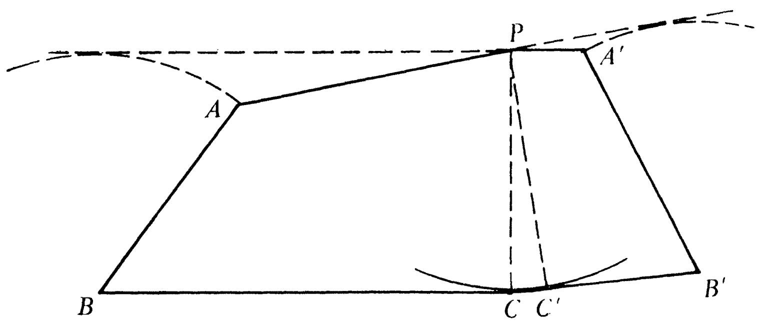


FIG. 3 Path length is the same as the motion.

the motion is greater since the motion of the A end of the moving line segment AB is from A to D', back to D, and then to A'. The arc DD' is traced three times. Hence, the total motion in Figure 4 is greater than the motion in Figure 3.

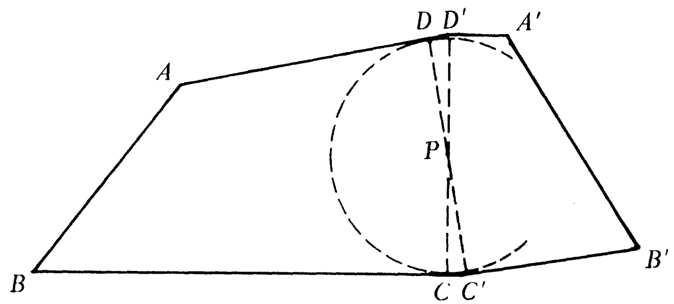


FIG. 4. Same path length as Fig. 3, but longer motion since part of path is retraced.

3. The restricted problem. If retracing of a point or an arc is not permitted, then a loop in the path is also not permitted. With this restriction, the length of the path is equal to the length of the motion. There may be one or more pivot points at which one end of the moving line is stationary, while the other end traces the arc of a circle. Then, the minimum motion is also the minimum path, and it may be obtained by the use of a mechanical analogy and elementary statics, without the use of the calculus of variations.

4. The restricted problem with no crossed paths. Consider the case, shown in Figure 3, in which the straight line AA' does not cross the line BB'. Let P be a pivot point where the A end is stationary, while the B end traces the arc CC'. Consider a taut string through $APA'B'C'$, along the arc $C'C$, and then through B and back to A. Subject to the restraints, the tension in the string will force the total length to be minimized. If the rigid body which includes the sector PCC', of unit radius, is in equilibrium with the string under uniform tension, then the resultant of the equal tensions at P is along the bisector of the angle at P, and it must be equal and opposite to the resultant of the tensions at C and C'. The points P, C, and C' are determined by the following construction. With B and B' as centers, draw arcs of unit radius. From A and A', draw tangents to these arcs. Their intersection is P. With P as a center, and with unit radius, draw an arc. Draw tangents BC and $B'C'$ to this arc. Then, AP and $C'B'$ are parallel, separated by unit distance. Similarly, PA' and BC are parallel, separated by unit distance. The extended lines AP, PA', BC, $C'B'$ make a rhombus. The sum of the equal tensions along AP and PA' is a resultant along the bisector of angle APA' (a diagonal of the rhombus), and it is equal and opposite to the resultant of the equal tensions along BC and $C'B'$.

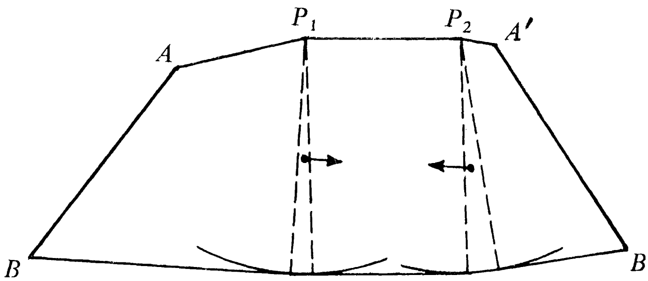


FIG. 5. Two pivot points, unstable.

If two pivot points P_1 and P_2 are used as shown in Figure 5, then the tensions in the string will force the two turning sectors together until they coincide to form Figure 3. In general, if any finite number of pivot points are used, the resultant of the tensions will force the sectors together until they coincide. From these cases, we can conclude that the method of Figure 3 gives the shortest possible motion for the restricted problem with no crossed paths, whenever the solution of Figure 1 cannot be used because of the given initial conditions.

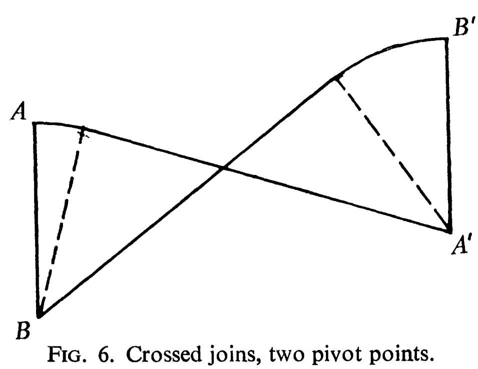


FIG. 6. Crossed joins, two pivot points.

5. The restricted problem with crossed paths. Consider the case, shown in Figure 6, in which the segment AA' crosses the segment BB'. Pivot points must be used near the terminal positions of the moving line unless one of the angles at the terminal line is a right angle or greater. In Figure 7, a pivot point is not needed at $A'B'$ since the angle at B' is obtuse. If the pivot centers had been taken at positions other than the terminal points, the tensions in the string would force the turning sectors to the terminal positions. The first selection of a turning arc must be at the largest acute angle in order to minimize the arc, and hence, to minimize the path.

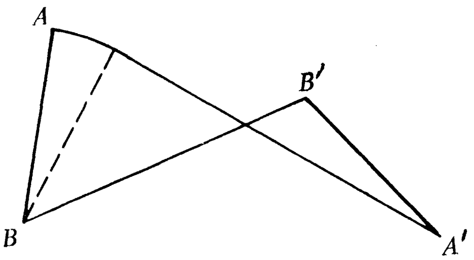


FIG. 7. Crossed joins, one pivot point.

6. The restricted problem with crossed joins and close terminal positions. In each of the foregoing cases, the motion is composed of a combination of rotations and translations. Figure 8, however, shows a case in which AA' crosses BB', and the positions AB and $A'B'$ are so close that tangents cannot be drawn to the turning arcs as in Figures 6 and 7. Then, it is necessary to use pure rotation of the line AB about a center P which is at the intersection of the perpendicular bisectors of AA' and BB'.

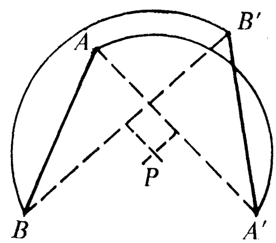


FIG. 8. Crossed joins, pure rotation.

A PRINCIPAL IDEAL RING THAT IS NOT A EUCLIDEAN RING

JACK C. WILSON, University of North Carolina at Asheville.

1. Introduction. In introductory algebra texts it is commonly proved that every Euclidean ring is a principal ideal ring. It is also usually stated that the converse is false, and the student is often referred to a paper by T. Motzkin [1]. Unfortunately, this reference does not contain all of the details of the counterexample, and it is not easy to find the remaining details from the references given in Motzkin's paper. The object of this article is to present the counterexample in complete detail and in a form that is accessible to students in an undergraduate algebra class.

Not all authors use precisely the same definitions for these two types of rings. Throughout this paper the following definitions will hold.

DEFINITION 1. *An integral domain R is said to be a Euclidean ring if for every $x \neq 0$ in R there is defined a nonnegative integer $d(x)$ such that:*

(i) *For all x and y in R, both nonzero, $d(x) \leqq d(xy)$.*

(ii) *For any x and y in R, both nonzero, there exist z and w in R such that $x = zy + w$ where either $w = 0$ or $d(w) < d(y)$.*

DEFINITION 2. *An integral domain R with unit element is a principal ideal ring if every ideal in R is a principal ideal; i.e., if every ideal A is of the form $A = (x)$ for some x in R.*

The ring, R, to be considered is a subset of the complex numbers with the usual operations of addition and multiplication:

$$R = \{a + b(1 + \sqrt{-19})/2 \mid a \text{ and } b \text{ are integers}\}.$$

It is elementary to show that R is an integral domain with unit element. The purpose of this article then is to show that R is a principal ideal ring, but that it is impossible to define a Euclidean norm on R so that with respect to that norm R is a Euclidean ring.

2. The ring is a principal ideal ring. In R there is the usual norm, $N(a + bi) = a^2 + b^2$, which has the property that $N(xy) = N(x)N(y)$ for all complex numbers x and y. In R this norm is always a nonnegative integer. The essential theorem for this part of the example is due to Dedekind and Hasse, and the proof is taken from [**2**, p. 100].

THEOREM 1. *If for all pairs of nonzero elements x and y in R with $N(x) \geqq N(y)$, either $y \mid x$ or there exist z and w in R with $0 < N(xz - yw) < N(y)$, then R is a principal ideal ring.*

Proof. Let $A \neq (0)$ be an ideal in R. Let y be an element of A with minimal nonzero norm, and let x be any other element of A. For all z and w in R, $xz - yw$ is in A so that either $xz - yw = 0$ or $N(xz - yw) \geqq N(y)$. Hence the assumed conditions on R require that $y \mid x$; i.e., $A = (y)$.

The ring R under consideration will now be shown to satisfy the hypotheses of Theorem 1. Observe that $0 < N(xz - yw) < N(y)$ if and only if $0 < N[(x/y)z - w] < 1$. Given x and y in R, both nonzero and $y \nmid x$, write x/y in the form $(a + b\sqrt{-19})/c$ where a, b, c are integers, $(a, b, c) = 1$, and $c > 1$. First of all, assume that $c \geqq 5$. Choose integers d, e, f, q, r such that $ae + bd + cf = 1$, $ad - 19\,be = cq + r$, and $|r| \leqq c/2$. Set $z = d + e\sqrt{-19}$ and $w = q - f\sqrt{-19}$. Thus,

$$\begin{aligned}(x/y)z - w &= (a + b\sqrt{-19})(d + e\sqrt{-19})/c - (q - f\sqrt{-19}) \\ &= r/c + \sqrt{-19}/c.\end{aligned}$$

This complex number is not zero and has norm $(r^2 + 19)/c^2$, which is less than 1 since $|r| \leqq c/2$ and $c \geqq 5$. The only case that is not immediately obvious is $c = 5$, but then $|r| \leqq 2$ so that $r^2 + 19 \leqq 23 < c^2$.

The remaining possibilities are $c = 2, 3$, or 4. Consider these in order:

(i) If $c = 2$, $y \nmid x$ and $(a,b,c) = 1$ imply that a and b are of opposite parity. Set $z = 1$ and $w = [(a-1) + b\sqrt{-19}]/2$ which are elements of R. Thus, $(x/y)z - w = 1/2 \neq 0$ and has norm less than 1.

(ii) If $c = 3$, $(a,b,c) = 1$ implies that $a^2 + 19b^2 \equiv a^2 + b^2 \not\equiv 0 \pmod 3$. Let $z = a - b\sqrt{-19}$ and $w = q$ where $a^2 + 19b^2 = 3q + r$ with $r = 1$ or 2. Thus, $(x/y)z - w = r/3 \neq 0$ and has norm less than 1.

(iii) If $c = 4$, a and b are not both even. If they are of opposite parity, $a^2 + 19b^2 \equiv a^2 - b^2 \not\equiv 0 \pmod 4$. Let $z = a - b\sqrt{-19}$ and $w = q$, where $a^2 + 19b^2 = 4q + r$ with $0 < r < 4$. Thus, $(x/y)z - w = r/4 \neq 0$ and has norm less than 1. If a and b are both odd, $a^2 + 19b^2 \equiv a^2 + 3b^2 \not\equiv 0 \pmod 8$. Let $z = (a - b\sqrt{-19})/2$ and $w = q$, where $a^2 + 19b^2 = 8q + r$ with $0 < r < 8$. Thus, $(x/y)z - w = r/8 \neq 0$ and has norm less than 1.

This completes the proof that R is a principal ideal ring.

3. The ring is not a Euclidean ring. This part of the counterexample is taken from [1]. The material is repeated and slightly elaborated here in order to give a self-contained result accessible to an undergraduate class. As with the previous section the results are stated within the context of the ring R under consideration, but the theorem applies to more general integral domains. Throughout this section R_0 will denote the set of nonzero elements of R.

DEFINITION 3. *A subset P of R_0 with the property $PR_0 \subset P$; i.e., xy is an element of P for all x in P and y in R_0, is called a product ideal of R. (Notice that R_0 is a product ideal.)*

DEFINITION 4. *If S is a subset of R, the derived set of S, denoted by S', is defined by $S' = \{x \in S \mid y + xR \subset S$, for some y in $R\}$.*

LEMMA 1. *If S is a product ideal, then S' is a product ideal.*

Proof. If x is in S', then x is in S and there exists y in R such that $y + xR \subset S$. Let z be in R_0. Since S is a product ideal and x is in S, xz is in S. Further, $y + (xz)R \subset y + xR \subset S$. This shows that $S'R_0 \subset S'$; i.e., S' is a product ideal.

LEMMA 2. *If $S \subset T$, then $S' \subset T'$.*

Proof. If x is in S', then x is in S and hence in T, and there exists a y in R such that $y + xR \subset S \subset T$. Therefore, x is in T', and $S' \subset T'$.

THEOREM 2. *If R is a Euclidean ring, then there exists a sequence, $\{P_n\}$, of product ideals with the following properties:*

(i) $R_0 = P_0 \supset P_1 \supset P_2 \supset \cdots \supset P_n \supset \cdots$,

(ii) $\cap P_n = \varnothing$,

(iii) $P_n' \subset P_{n+1}$, *for each n, and*

(iv) *For each n, $R_0^{(n)}$, the nth derived set of R_0, is a subset of P_n.*

Proof. Let the Euclidean norm in R be symbolized by $d(x)$ for x in R_0. For each nonnegative integer n, define $P_n = \{x \in R_0 \mid d(x) \geqq n\}$. This defines the sequence which obviously has properties (i) and (ii). Suppose that x is in P_n and y is in R_0. $d(xy) \geqq d(x) \geqq n$ which implies that xy is in P_n. This shows that $P_n R_0 \subset P_n$; i.e., for each n, P_n is a product ideal.

For property (iii) let x be in P'_n; i.e., x is in P_n and there exists a y in R such that $y + xR \subset P_n$. Applying the Euclidean algorithm, there exist elements q and r in R with $y = xq + r$ and $r = 0$ or $d(r) < d(x)$. Hence, $r = y + x(-q)$ is in $y + xR \subset P_n$, which implies that $d(r) \geqq n$, and in turn, $d(x) > d(r) \geqq n$, so that $d(x) \geqq n + 1$ and x is in P_{n+1}. This proves property (iii) $P'_n \subset P_{n+1}$.

For property (iv), clearly $R_0 = P_0$ and application of (ii) gives $R'_0 = P'_0 \subset P_1$. Assuming that $R_0^{(n)} \subset P_n$, Lemma 2 and (iii) yield $R_0^{(n+1)} \subset P'_n \subset P_{n+1}$. By induction, (iv) is proved.

COROLLARY. *If $R'_0 = R''_0 \neq \varnothing$, then R is not a Euclidean ring.*

Proof. The hypotheses of the corollary imply that for all n, $R_0^{(n)} = R'_0$. If R is a Euclidean ring, the theorem would require $R'_0 = \bigcap R_0^{(n)} \subset \bigcap P_n = \varnothing$.

This corollary is now used to show that R is not a Euclidean ring. First R'_0 is determined. If x is a unit in R, say $xy = 1$, and z is an element of R, $z + x(-yz) = 0$ is not in R_0. This shows that units are not in R'_0. If x is not a unit in R, then using $z = -1$, $z + xy \neq 0$ for all y in R, which shows that if x is not zero and not a unit, x is in R'_0. Altogether, R'_0 is precisely the set of elements of R that are neither units nor zero. Notice that the only units of our example R are 1 and -1. Next, in order to determine the elements of R''_0, it is convenient to use the following terminology:

DEFINITION 5. *An element x of R'_0 is said to be a side divisor of y in R provided there is a z in R that is not in R'_0 such that $x \mid (y + z)$. An element x of R_0 is a universal side divisor provided that it is a side divisor of every element of R.*

If x is in R''_0, then x is in R'_0 and there is a y in R such that $y + xR \subset R'_0$; i.e., x never divides $y + z$ if z is zero or a unit. Thus, x is not a side divisor of y, and therefore, not a universal side divisor. Conversely, if x is not in R''_0, and is in R'_0, then for every y in R there exists a w in R with $y + xw$ not in R'_0; i.e., $y + xw$ is zero or a unit, and therefore, x is a side divisor of y. Since this holds for every y in R, x is a universal side divisor. Together, these two arguments show that R''_0 is the set R'_0 exclusive of the universal side divisors. If it can now be shown that R has no universal side divisors, this will show that $R'_0 = R''_0 \neq \varnothing$, and the corollary will complete the proof that R is not a Euclidean ring.

A side divisor of 2 in R must be a nonunit divisor of 2 or 3. In R, 2 and 3 are irreducible, and therefore, the only side divisors of 2 are 2, -2, 3, and -3. On the other hand, a side divisor of $(1 + \sqrt{-19})/2$ must be a nonunit divisor of $(1 + \sqrt{-19})/2$, $(3 + \sqrt{-19})/2$, or $(-1 + \sqrt{-19})/2$. These elements of R have norms of 5, 7, and 5, respectively, while the norms of 2 and 3 and their associates are 4 and 9, respectively. As a result, no side divisor of 2 is also a side divisor of

$(1+\sqrt{-19})/2$, and there are no universal side divisors in R. All of the details of the counterexample are complete.

References

1. T. Motzkin, The Euclidean algorithm, Bull. Amer. Math. Soc., 55 (1949) 1142–1146.
2. H. Pollard, The Theory of Algebraic Numbers, Carus Monograph 9, MAA, Wiley, New York, 1950.

THE U.S.A. MATHEMATICAL OLYMPIAD

Sponsored by the Mathematical Association of America, the first U.S.A. Mathematical Olympiad was held on May 9, 1972. One hundred students participated. The eight top ranking students were: James Saxe, Albany, N.Y.; Thomas Hemphill, Sepulveda, Calif.; David Vanderbilt, Garden City, N.Y.; Paul Harrington, Central Square, N.Y.; Arthur Rubin, West Lafayette, Ind.; David Anick, New Shrewsbury, N.J.; Steven Raher, Sioux City, Iowa; James Shearer, Livermore, Calif. A detailed report on the Olympiad including the problems and solutions will appear in the March 1973 issue of the American Mathematical Monthly.

The second U.S.A. Mathematical Olympiad will be administered on Tuesday, May 1, 1973. Participation is by invitation only. For further particulars, please contact the Chairman of the U.S.A. Mathematical Olympiad Committee, Dr. Samuel L. Greitzer, Mathematics Department, Room 212, Smith Hall, Rutgers University, Newark, N.J. 07102.

GLOSSARY

KATHARINE O'BRIEN, Portland, Maine

Campus disorder

Skew quadrilateral-quadrangle –
extreme polarization
demonstration.

Generation gap

Two-parameter family –
negative orientation
to communication.

Heart transplant

Removable discontinuity –
binary correlation
operation.

Nylon tires

Synthetic substitution –
translation by rotation
transportation.

Popcorn

Iterated kernels –
onto magnification
transformation.

Suburb

Deleted neighborhood –
little inclination
to integration.

ANOTHER PROOF OF A THEOREM OF NIVEN

KENNETH S. WILLIAMS, Carleton University, Ottawa

Niven [3] has proved that the gaussian integer $a + 2bi$ (a, b integers) is the sum of two squares of gaussian integers if and only if $(1 + i)^3 \not\parallel a + 2bi$. (If w ($\neq 0$) and z are gaussian integers such that $w^k \mid z$, $w^{k+1} \nmid z$ for some integer $k \geqq 1$ we write $w^k \parallel z$.) Simple proofs of this result have been given recently by Leahey [1] and Mordell [2]. Here is another simple proof.

We begin by showing that if $(1 + i)^3 \not\parallel a + 2bi$ then $a + 2bi$ is the sum of two squares of gaussian integers. If a is odd, so that $1 + i \nmid a + 2bi$, we have

$$a + 2bi = \left(\frac{(a+1)}{2} + bi\right)^2 + \left(b - \frac{(a-1)}{2}i\right)^2.$$

If a is even we have $(1 + i)^2 \mid a + 2bi$. If $(1 + i)^2 \parallel a + 2bi$, say $a + 2bi = (1 + i)^2(c + di)$, where $c + d$ odd, then

$$a + 2bi = \left\{\left(\frac{c-d+1}{2}\right) + i\left(\frac{c+d+1}{2}\right)\right\}^2 + \left\{\left(\frac{c+d-1}{2}\right) + i\left(\frac{-c+d+1}{2}\right)\right\}^2.$$

If $(1 + i)^4 \mid a + 2bi$, say $a + 2bi = (1 + i)^4(e + fi)$, then we have

$$a + 2bi = ((e-1) + fi)^2 + (f - (e+1)i)^2.$$

Finally suppose $(1 + i)^3 \parallel a + 2bi$, say $a + 2bi = (1 + i)^3(g + hi)$, where $g + h$ is odd. We show that $a + 2bi$ is not the sum of two squares of gaussian integers, for if $a + 2bi = (a_1 + b_1 i)^2 + (a_2 + b_2 i)^2$ then

$$(1 + i)^3(g + hi) = \{(a_1 + b_2) - (a_2 - b_1)i\}\{(a_1 - b_2) + (a_2 + b_1)i\},$$

and so on multiplying both sides by their complex conjugates we obtain

$$2^3(g^2 + h^2) = \{(a_1 + b_2)^2 + (a_2 - b_1)^2\}\{(a_1 - b_2)^2 + (a_2 + b_1)^2\},$$

which a simple parity argument shows to be impossible as the left hand side is $\equiv 8 \pmod{16}$ yet the right hand side is $\equiv 0, 1, 4, 5, 9, 13 \pmod{16}$. This completes the proof.

References

1. W. J. Leahey, A note on a theorem of I. Niven, Proc. Amer. Math. Soc., 16 (1965) 1130–1131.
2. L. J. Mordell, The representation of a gaussian integer as a sum of two squares, this MAGAZINE, 40 (1967) 209.
3. I. Niven, Integers of quadratic fields as sums of squares, Trans. Amer. Math. Soc., 48 (1940) 405–417.

CLASS NOTES ON SERIES RELATED TO THE HARMONIC SERIES

MILTIADES S. DEMOS, Villanova University

We know that $\Sigma_1^\infty (1/k)$ diverges and $\Sigma_1^\infty ((-1)^{k-1}/k) = \ln 2$. We consider some rearrangements of the alternating series. All limits are as $n \to \infty$.

LEMMA. *If $a > b > 0$ are integers,* $\lim \Sigma_{nb+1}^{na} (1/k) = \ln(a/b)$.

Proof. $$\frac{1}{nb+1} + \frac{1}{nb+2} + \cdots + \frac{1}{na} = \frac{1}{n}\left[\frac{1}{b+\frac{1}{n}} + \frac{1}{b+\frac{2}{n}} + \cdots + \frac{1}{a}\right].$$

By the definition of an integral $\lim \Sigma_{nb+1}^{na} (1/k) = \int_b^a (dx/x) = \ln(a/b)$. In particular $\lim \Sigma_n^{na} (1/k) = \ln a$.

Suppose in the series for $\ln 2$, we take a positive terms followed by b negative terms and alternate this way.

THEOREM 1. *Let*

$$S_n = \left(\frac{1}{1} + \frac{1}{3} + \cdots + \frac{1}{2a-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2b}\right)$$
$$+ \left(\frac{1}{2a+1} + \cdots + \frac{1}{4a-1}\right) - \left(\frac{1}{2b+2} + \cdots + \frac{1}{4b}\right) + \cdots$$
$$+ \left(\frac{1}{(n-1)a+1} + \frac{1}{(n-1)a+3} + \cdots + \frac{1}{na-1}\right) - \left(\frac{1}{2(n-1)b+2} + \cdots + \frac{1}{2nb}\right).$$

Then $\lim S_n = \ln 2 + \frac{1}{2}\ln(a/b)$.

Proof (when $a > b$).

$$S_n = \sum_1^{na} \frac{(-1)^{k-1}}{k} + \sum_{nb+1}^{na} \frac{1}{2k}.$$

Since the last sum equals $\frac{1}{2}\Sigma_{nb+1}^{na} (1/k)$ the theorem follows. A similar proof can be given for $a < b$.

THEOREM 2. *In the harmonic series replace the kath term by* $-(a-1)/(ka)$. *Then the series converges to* $\ln a$.

Proof. Set

$$S_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{a-1} - \frac{a-1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{2a-1} - \frac{a-1}{2a}$$
$$+ \cdots + \frac{1}{(n-1)a+1} + \cdots + \frac{1}{na-1} - \frac{a-1}{na}.$$

Then

$$S_n = \sum_1^{na} \frac{1}{k} - \sum_1^{n} \frac{1}{ka} - \sum_1^{n} \frac{a-1}{ka} = \sum_1^{na} \frac{1}{k} - \sum_1^{n} \frac{1}{k} = \sum_{n+1}^{na} \frac{1}{k} \to \ln a.$$

It is easily seen that the sequence of partial sums of the series has the same limit as S_n.

We know $\sum_{k=1}^{\infty} (1/k^p)$ converges if and only if $p > 1$. Suppose that we sum the series for all k that can be written without using the numeral 9. Suppose, for example, that the key for 9 on the typewriter is broken and we type all the natural numbers we can. Call the summation Σ'.

THEOREM 3. *Σ' $(1/k^p)$ converges if and only if* $p > \log_{10} 9 = .954+$.

Proof. There are (9–1) terms with $\dfrac{1}{10^p} < \dfrac{1}{k^p} \leqq \dfrac{1}{1^p}$

$$9^2 - 9 \text{ terms with } \frac{1}{100^p} < \frac{1}{k^p} \leqq \frac{1}{10^p}$$

$$9^n - 9^{n-1} \text{ terms with } \frac{1}{10^{np}} < \frac{1}{k^p} \leqq \frac{1}{10^{(n-1)p}}.$$

Then

$$\frac{8}{10^p} + \frac{8.9}{10^{2p}} + \frac{8.9^2}{10^{3p}} + \cdots < \Sigma' \frac{1}{k^p} < \frac{8}{1^p} + \frac{8.9}{10^p} + \frac{8.9^2}{10^{2p}} + \cdots$$

The ratio of both geometric series is $(9/10^p)$. Then the series converges if and only if $(9/10^p) < 1$, that is, $p > \log_{10} 9$.

We would obtain the same result if instead of 9, any other numeral were missing. If two numerals are missing, we can show that the series converges if and only if $p > \log_{10} 8$. In general, if we use exactly m numerals $(1 \leqq m \leqq 10)$, the series converges if and only if $p > \log_{10} m$, with the proviso that when we use only one numeral that numeral is not zero.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) *will be placed by the problem number to indicate that the proposer did not* *supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.*

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before July 1, 1973.

PROPOSALS

852. *Proposed by Richard A. Gibbs, Fort Lewis College, Colorado.*

If a $2k \times 2k$ checkerboard is given with two diagonal corners removed, it is well known that it cannot be covered by 2×1 dominoes because the removed corners are of the same color and a domino must cover one square of each color. (a) Is it possible to cover a $2k \times 2k$ checkerboard which has had one square of each color removed? (b) What about a $(2k+1) \times (2k+1)$ checkerboard which has had one square of the corner color removed?

853. *Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.*

It is a well-known theorem that all quadric surfaces which pass through seven given points will also pass through an eighth fixed point. (a) If the seven given points are $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(2,0,0)$, $(1,1,0)$, $(1,0,1)$ and $(1,1,1)$, determine the eighth fixed point. (b) Determine the eighth fixed point explicitly as a function of the seven general given points (x_i, y_i, z_i), $i = 1, 2, 3 \ldots 7$.

854. *Proposed by John D. Baum, Oberlin College, Ohio.*

Let $N = x^4 + 4a^4$ where x and a are integers, then N is composite unless $x = a = \pm 1$.

855. *Proposed by Romae J. Cormier, Northern Illinois University, De Kalb, Illinois.*

Let T and T' be two Pythagorean triangles. If θ and θ' are any acute angles of these triangles respectively such that $\theta \neq \theta'$, then show that the right triangle T'' which has an acute angle $\theta + \theta'$ or $\pi - \theta - \theta'$ is Pythagorean.

***856.** *Proposed by Leonard Gallagher, University of Colorado.*

For what values of x and y is the following logarithmic inequality valid:

$$\frac{\log^2 x \log^2 y}{\log^2 x + \log^2 y} < \log^2 \left(\frac{xy}{x+y}\right).$$

857. *Proposed by Marlow Sholander, Case Western Reserve University.*

Find a function $y = f(x)$ such that:

(i) $f(0) = 2$, $f(2) = 1$, $f(4) = 2$,
(ii) $f(x)$ is differentiable on $0 \leqq x \leqq 4$,
(iii) There is an area-length equality

$$\int_0^x y\,dx = \int_0^x \sqrt{1+(y')^2}\,dx.$$

858. *Proposed by David Singmaster, Polytechnic of the South Bank, London, England.*

In the November 1970 issue, Eric Langford has obtained the probability that three points chosen at random in a $1 \times L$ rectangle form an obtuse triangle. Consider two points chosen at random in the unit interval. It is not difficult to see that the probability of the three segments forming a triangle is $\frac{1}{4}$. What is the probability that the resulting triangle is obtuse?

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q557. Prove that $x^4 + 2y^4 + 3z^4 - 4w^4 = 7$ has no solution in integers.

[*Submitted by Erwin Just*]

Q558. It is known that if a ray of light is reflected off three successive faces of a trirectangular corner mirror, the final direction of the ray is parallel but opposite to that of the incoming ray. Show that the same property holds more generally for n successive reflections off the n faces of an n-rectangular corner mirror in E^n.

[*Submitted by Murray S. Klamkin*]

Q559. If $a_{n+1} = 5a_n + \sqrt{24a_n^2 - 1}$, $n = 0, 1, 2 \ldots$ and $a_0 = 0$, show that the sequence $\{a_n\}$ is always integral.

[*Submitted by Murray S. Klamkin*]

Q560. In the set R^+ of positive real numbers, consider the usual binary operations $+$ and $\cdot$. Is $(R^+, \cdot, +)$ a ring?

[*Submitted by Jürg Rätz, Switzerland*]

Q561. The arithmetic mean of the twin primes 3 and 5 is the square integer 4. Are there other twin primes with a square arithmetic mean?

[*Submitted by Charles W. Trigg*]

(Answers on page 54)

SOLUTIONS

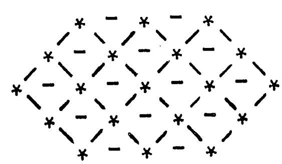
Late Solutions

James H. Boersma, Granville, Michigan: **817**; *Derrill J. Bordelon, Naval Underwater Systems Center, Rhode Island:* **818, 819**; *Helen Engebretson, Brookings, South Dakota:* **821**; *M. G. Greening, University of New South Wales, Australia:* **817, 819, 821, 822, 823**; *Bryan V. Hearsey, Lebanon Valley College, Pennsylvania:* **817**; *Dean Hickerson, Davis, California:* **823**; *Thomas J. Hofman, Wanewac, Wisconsin:* **817**; *Shiv Kumar and Miss Nirmal (jointly), Ohio University:* **817, 818**; *M. S. Krishnamoorthy, Kanpur, India:* **817**; *Larry Masselink, Fremont Christian Schools, Michigan:* **820**; *Edwin P. McCravy, Midlands Technical Education Center, South Carolina:* **817**; *Gordon Miller, Wisconsin State University, Stevens Point:* **817, 821**; *Rev. Bernard J. Portz, Creighton University, Nebraska:* **817**; *Fred Pooley, Parkston, South Dakota:* **817**; *Jack S. Selver, Peter Csontos, Chris Morgan (jointly), California State College at Hayward:* **817**; *Roland F. Smith, Russell Sage College, New York:* **817**; *Ron Soenkssen, Pierce, Nebraska:* **817**; *Charles H. Stansbury, Western Michigan University:* **817**; *and Phil Tracy, Liverpool, New York:* **821**.

A Well-Known Magic Hexagon

824. [March, 1972] *Proposed by Paul S. Lemke, Rensselaer Polytechnic Institute.*

Place the integers 1 through 19 so as to form a "magic hexagon": the sums in each of the fifteen ways indicated are all the same:



Solution by Charles W. Trigg, San Diego, California.

The desired third order (the number, n, of elements on each side is 3) row-magic hexagon (there are other types) with every one of the 15 rows having the magic sum of 38 is:

15 13 10
14 8 4 12
9 6 5 2 16
11 1 7 19
18 17 3

A mirror image of this hexagon by T. Vickers appeared without comment on Page 29 of the December 1958 Mathematical Gazette. It had been developed independently by Clifford W. Adams, as reported by Martin Gardner on Page 116 of the August 1963 Scientific American. In *A unique magic hexagon*, Recreational Mathematics Magazine (January 1964), Pages 40–43, I proved that this is the only third order row-magic hexagon and that none exists for other values of n. The uniqueness was confirmed by W. M. Daly and by G. W. Anderson using computers, and by Eduardo Esperón using simultaneous equations but no computer. This magic hexagon also appears on Page 101 of Joseph S. Madachy's book, *Mathematics on Vacation* (1966).

A related article is my *Hetro-hexagons of the second order*, School Science and Mathematics (February 1966), Pages 216–217.

References and solutions also provided by Jeffrey H. Baumwell, Whitestone, New York; Mannis Charosh, Brooklyn, New York; Edward T. Frankel, Schenectady, New York; G. A. Heuer, Concordia College, Minnesota; V. F. Ivanoff, San Carlos, California; Sam Kravitz, Cleveland Heights, Ohio; Paul S. Lemke, Rensselaer Polytechnic Institute; Meredith A. Mackierman, Georgia Southern College; Otto Mond, Suffern, New York; Thomas E. Moore, Bridgewater State College, Massachusetts; Fred Pence, Harrisonburg, Virginia; Sally Ringland, Clarion State College, Pennsylvania; S.O. Schachter, Philadelphia, Pennsylvania; Jan B. Schipmalder, University of California at San Diego; David Singmaster, Polytechnic of the South Bank, London, England; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

Edward T. Frankel commented as follows:

This problem and the solution below have a long history. Clifford W. Adams came across the problem in 1910. He worked on the problem by trial and error and after many years arrived at the solution which he transmitted to Martin Gardner, editor of *Mathematical Games* in the Scientific American.

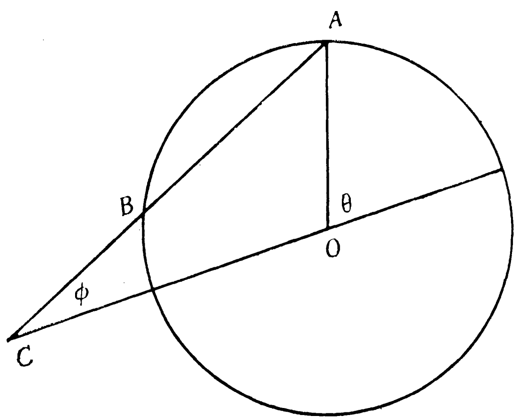
Gardner sent Adams' magic hexagon to Charles W. Trigg who by mathematical analysis found that it was unique, disregarding rotations and reflections.

Adams' result and Trigg's work were written up in the Scientific American, August 1963, page 116. Trigg did further research, then summarized known results and the history of the problem in his fascinating article *A unique magic hexagon*, in Recreational Mathematics magazine, January-February 1964, pages 40–43.

A Relation Between Two Angles

825. [March, 1972] *Proposed by Henry W. Gould, West Virginia University.*

One method of trisecting an angle uses compasses (to describe a circle with



radius R) and a straight edge with a distance between two marks equal to R. In the figure $CB = R$ with the result that $\phi = \theta/3$.

An incorrect variant of this method uses a straight edge with arbitrary markings such that $CB = BA = x$. In this case establish the relationship between ϕ and θ and determine whether trisection is ever achieved. Extend the discussion to the case $CB = m(BA)$.

I. *Solution by Vaclav Konečny, Jarvis Christian College, Texas.*

If $CB = BA = x$ then $R \sin\theta = 2x \sin\phi$ and $x = 2R \cos(\theta - \phi)$. Eliminating x and R we get $\sin\theta = 4\cos(\theta - \phi)\sin\phi$, which is the required relation between ϕ and θ. The trisection is achieved if $\theta = 3\phi$. Thus substituting 3ϕ for θ into this equation we get $\sin 3\phi = 4 \cos 2\phi \sin\phi$. The solution of this equation in $\sin\phi$ is $\sin\phi = \pm\frac{1}{2}$ and $\sin\phi = 0$. Considering $0 < \phi < \pi/2$, $\phi = \pi/6$ and $\theta = \pi/2$. (Trisection can be achieved.) If $CB = m(BA)$ we get $R\sin\theta = (mx + x)\sin\phi$ and $x = 2R\cos(\theta - \phi)$ where we have put $BA = x$. Thus the relation between ϕ and θ is $\sin\theta = 2(m+1)\cos(\theta - \phi)\sin\phi$. Trisection is achieved if $\theta = 3\phi$ that is if $\sin 3\phi = 2(m+1)\cos 2\phi \sin\phi$. The solution of this equation is $\sin\phi = \pm\frac{1}{2}\{(2m-1)/m\}^{\frac{1}{2}}$. So the trisection can be achieved only if $m > \frac{1}{2}$ or $m < -\frac{1}{2}$.

II. *Solution by Charles W. Trigg, San Diego, California.*

In the figure, in triangle CAO, $2x/\sin(180° - \theta) = R/\sin\phi$;
in triangle CBO, $x/\sin(\theta - 2\phi) = R/\sin\phi$.

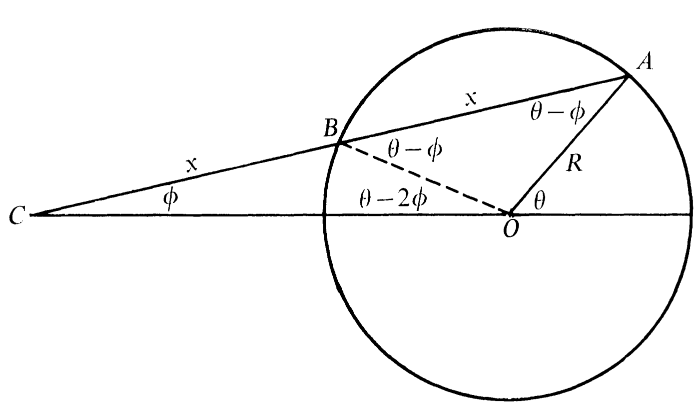
Thus $x/R = \sin\theta/2\sin\phi = \sin(\theta - 2\phi)/\sin\phi$.

Hence, if $\theta = 3\phi$, then $x = R$. This is the only trisection situation except the trivial, $x = 2R$, $\theta = 0 = 3\phi$.

Furthermore, $2x/\sin\theta = x/\sin(\theta - 2\phi)$. Whereupon, $2(\sin\theta\cos 2\phi - \cos\theta\sin 2\phi) = \sin\theta$ and $\tan\theta = \sin\theta/\cos\theta = \sin 2\phi/(2\cos 2\phi - 1)$.

If $x = CB = m(BA)$, trisection still occurs when $x = R$, and in general

$$\tan\theta = \sin 2\phi/[(1+m)\cos 2\phi - 1].$$



Also solved by Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Ralph Jones, University of Massachusetts; Vaclav Konecny, Hawkins, Texas; K. R. S. Sastry, Makele, Ethiopia; Ron Soenksen, Pierce, Nebraska; Steven Szabo, University of Illinois; Phil Tracy, Liverpool, New York; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

Minimal Stock Purchase

826. [March, 1972] *Proposed by F. D. Parker, St. Lawrence University.*

Two men, A and B, purchase stock in the same company at times $t_1, t_2, t_3, \ldots t_n$, when the price per share is respectively $p_1, p_2, p_3, \ldots, p_n$. Their methods of investment are different, however: A purchases x shares each time, whereas B invests P dollars each time (we assume it is possible to purchase fractional shares). Show that unless $p_1 = p_2 = \ldots = p_n$, the average cost per share for B is less than the average cost per share for A.

I. *Solution by Stephen K. Park, NASA, Langley Research Center, Hampton, Virginia.*

The solution to this problem is a direct, but interesting, application of the Schwarz nequality. For, if one defines the real n-vectors a, b with components $a_i = p_i^{-1/2}$ and $b_i = p_i^{1/2}$ respectively, then from $(a,b)^2 \leqq (a,a) \cdot (b,b)$ we obtain

$$n^2 \leqq \left(\sum_{i=1}^{n} \frac{1}{p_i} \right) \left(\sum_{i=1}^{n} p_i \right)$$

with equality iff $p_1 = p_2 = \ldots = p_n$. Equivalently,

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{p_i}} \leqq \frac{1}{n} \sum_{i=1}^{n} p_i$$

and the result follows since the left and right hand sides of this inequality represent the average cost per share for B and A respectively.

II. *Solution by J. R. Hanna, University of Wyoming.*

(1) Average cost per share for A:

$$[1/(xn)] \left[x \sum_{j=1}^{n} p_j \right] = \frac{1}{n} \sum_{j=1}^{n} p_j = \text{A.M.}$$

(2) Average cost per share for B:

$$[nP] / \sum_{j=1}^{n} [P/p_j] = n / \sum_{j=1}^{n} [1/p_j] = \text{H.M.}$$

The average for A is an arithmetic mean (A.M.); and for B a harmonic mean (H.M.) for the items $p_j (j = 1, \ldots n)$. H.M. $\leqq$ A.M. (equality holds only if all sample values p_j are identical). The average for B is less than the average for A unless all p_j's are equal, then the two averages are equal.

Also solved by Donald Batman, MIT Lincoln Laboratory; Dermott A. Breault, Microsystems Technology Corporation, Burlington, Maine; J. L. Brown, Jr., Pennsylvania State University; Fred Dodd, University of South Alabama; Robert N. Eckert, Culver, Indiana; Ralph Garfield, The College of Insurance, New York; Michael Goldberg, Washington, D. C.; Richard A. Groeneveld, Iowa State

University; J. R. Hanna, University of Wyoming; James C. Hickman, University of Iowa; Thomas W. Hill, Jr., Purdue University; Steven Jamke, University of California, Berkeley;Allan W. Johnson, Jr., Defense Communications Agency, Washington, D. C.; Ralph Jones, University of Massachusetts; Lew Kowarski, Morgan State College, Maryland; Norbert J. Kuenzi, Oshkosh, Wisconsin; Otto Mond, Suffern, New York; Roger H. Marty, Cleveland State University, Ohio; E. B. McLeod, California State University, Long Beach; Lester Meckler, Levittown, New York; John E. Prussing, University of Illinois; Tim Robertson, University of Iowa; Rena Rubenfeld, New York City Community College; K. R. S. Sastry, Makele, Ethiopia; David R. Stone, Georgia Southern College; Phil Tracy, Liverpool, New York; Ronald H. Whiffen, Flushing, New York; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

Triangle Area Ratios

827. [March, 1972] *Proposed by V. F. Ivanoff, San Carlos, California*

Prove that in a triangle with sides a, b, c and angles α, β, γ:

$$\frac{\cot\alpha}{b^2+c^2-a^2}=\frac{\cot\beta}{c^2+a^2-b^2}=\frac{\cot\gamma}{a^2+b^2-c^2}$$

and find the geometric interpretation of the ratios.

Solution by Leon Bankoff, Los Angeles, California.

$$\frac{\cot\alpha}{b^2+c^2-a^2}=\frac{\cot\alpha}{2bc\cos\alpha}=\frac{1}{2bc\sin\alpha}=\frac{1}{4\Delta}$$

$$\frac{\cot\beta}{c^2+a^2-b^2}=\frac{\cot\beta}{2ac\cos\beta}=\frac{1}{2ac\sin\beta}=\frac{1}{4\Delta}$$

$$\frac{\cot\gamma}{a^2+b^2-c^2}=\frac{\cot\gamma}{2ab\cos\gamma}=\frac{1}{2ab\sin\gamma}=\frac{1}{4\Delta}$$

The three given ratios are equal since each is equivalent to the reciprocal of 4Δ, where Δ is the area of the triangle. Another way of expressing this ratio is R/abc, where R is the circumradius of the triangle.

Also solved by Olowoye S. Adegboye, Ahmadu Bello University, Kano, Nigeria; Merrill Barnebey, Holmen, Wisconsin; M. T. Bird, California State University, San Jose; V. S. Blanco, University of South Alabama; Dermott A. Breault, Microsystems Technology Corporation, Burlington, Massachusetts; Richard L. Breisch, Royersford, Pennsylvania; Elio E. del Canal, Attleboro, Massachusetts; Mannis Charosh, Brooklyn, New York; Robert W. Chilcote, Bedford High School, Bedford, Ohio; Fred Dodd, University of South Alabama; Ragnar Dybvik, Tingvall, Norway; Robert N. Eckert, Culver, Indiana; Gabriel V. Ferrer, University Autonoma de Baja California, Mexico; Herta T. Freitag, Hollins, Virginia; Ralph Garfield, College of Insurance, New York; Michael Goldberg, Washington, D. C.; H. W. Gould, West Virginia University; J. R. Hanna, University of Wyoming; Robert S. Hatcher, Santa Catalina School, Monterey, California;Ralph Jones, University of Massachusttes; Dan Kellman, Philadelphia, Pennsylvania; Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan; Vaclav Konecny, Jarvis Christian College, Texas; J. D. E. Konhauser, Macalester College, Minnesota; V. Linis, University of Ottawa, Canada; Lester Meckler, Levittown, New York; Virginia T. Merrill, Solon, Maine; Gabriel Rosenberg, Pace College, New York;Rina Rubenfeld, New York City Community College; W. M. Sanders, Madison College; E. P. Starke, Plainfield, New Jersey; David R. Stone, Georgia Southern College; Robert A. Sutton, Jr., Thomas Jefferson High School, San Antonio, Texas;

Steven Szabo, University of Illinois; Phil Tracy, Liverpool, New York; Charles W. Trigg, San Diego, California; William Wernick, City College, New York; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

N-linked M-chains

828. [March, 1972] *Proposed by Warren Page, New York City Community College.*

Call an n-digit number $x = x_1x_2 \ldots x_n$ an n-linked m-chain if $x_1 + x_n = x_2 + x_{n-1} = x_3 + x_{n-2} \ldots = m$, with $x_{(m+1)/2} = m$ when n is odd. The number 25614 for example is a 5-linked 6-chain.

What is the largest natural number n such that for every n-digit number $x_1x_2 \ldots x_n, x_1 \neq x_n, \left| x_1x_2 \ldots x_n - x_nx_{n-1} \ldots x_1 \right|$ is a k-linked 9-chain, $k \leqq n$?

Can these concepts be extended further?

Solution by Robert N. Eckert, Culver, Indiana.

For $n = 2$, the difference $x_2x_1 - x_1x_2$ must be a multiple of 9 greater than 0 (if x_2 is greater than x_1), since x_2x_1 and x_1x_2 are both congruent to the same number $(x_1 + x_2)$ in modulo 9. Thus, it will either equal 9 itself, a 1-linked 9-chain, or some two digit number y_1y_2, which, being a multiple of 9, must have the sum of its digits, $y_1 + y_2$, equal to 9, making it a 2-linked 9-chain. Thus, $73 - 37 = 36$, $3 + 6 = 9$.

For $n = 3$, in taking the difference $x_1x_2x_3 - x_3x_2x_1$, we find that the rightmost subtraction will be negative if x_1 is greater than x_3, and therefore that a 1 must be borrowed from x_2. Then, the next to rightmost subtraction will be $(x_2 - 1) - x_2$ and will also require a borrowing to give final answer 9. Thus, the middle digit of the final answer will always be 9, and the total sum of the digits must be a multiple of 9, since the difference is a multiple of 9, as above. So the sum of the first and last digits of the answer will be a multiple of 9. This sum cannot be 0 and cannot be 18 or over (since the only two digits summing up so high are 9 and 9, and it is impossible for the difference of two three-digit numbers to be 999). Therefore, it is 9, and the difference is a 3-link 9-chain. Thus, $573 - 375 = 198$.

For $n = 4$, the counterexample $2201 - 1022 = 1179$ shows that it is not true for all 4-digit numbers. Thus, 3 is the highest such. These results are perfectly general: the proofs that the property holds for two and three digit numbers still hold for base twelve, substituting 11 for 9 throughout, or for base five, substituting 4 for 9, etc., and the counterexample 2201 for $n = 4$ exists in all bases (except, of course, for base two).

Also solved by Michael Goldberg, Washington, D. C.; Rina Rubenfeld, New York City Community College; David R. Stone, Georgia Southern College; Phil Tracy, Liverpool, New York; and the proposer.

Unique Representation of Integers

829. [March, 1972] *Proposed by John D. Baum, Oberlin College.*

It is well known that a positive integer can be written as the sum of consecutive integers if and only if it is not a power of two. If a positive integer is so expressible, its representation is not necessarily unique. For example,

$$15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5.$$

For integers of what form are their expressions as sums of positive consecutive integers unique?

Solution by E. P. Starke, Plainfield, New Jersey.

To be expressible uniquely as the sum of consecutive positive integers, a positive integer must be of the form $2^a p$, where a is a nonnegative integer and p is a prime.

This is a special case of the result of Problem E2225, American Mathematical Monthly, 1971, Page 199, where it is shown that the number of representations of a positive integer n as a sum of consecutive positive integers equals the number of odd divisors of n. This and similar problems are discussed also in Polya, *Mathematical Discovery*, V. II (1965) Problems 15.48, 15.49, 15.50.

Note. The first sentence of the Proposal should state "as the sum of consecutive *positive* integers." It is not otherwise true. E.g., $4 = (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4$.

Also solved by Merrill Barnebey, Holmen, Wisconsin; Donald Batman, MIT Lincoln Laboratory; Ruth E. Bauman, Kalamazoo, Michigan; Elio E. del Canal, Bishop Feehan High School, Attleboro, Massachusetts; Robert N. Eckert, Culver, Indiana; Richard A. Gibbs, Fort Lewis College, Colorado; Marc Glucksman, El Camino College, California; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Richard A. Groeneveld, Iowa State University; Ralph Jones, University of Massachusetts; Vaclav Konecny, Jarvis Christian College, Texas; Norbert J. Kuenzi and Bob Prielipp (jointly), University of Wisconsin at Oshkosh; V. Linis, University of Ottawa, Canada; Arthur Marshall, Madison, Wisconsin; John W. Milsom, Butler County Community College, Pennsylvania; John D. Moore, Niagara University, New York; Problem Solving Group, Bern, Switzerland; Marilyn Rodeen, San Francisco, California; The 3 S Group, New York, New York; David R. Stone, Georgia Southern College; Phil Tracy, Liverpool, New York; Kenneth M. Wilke, Topeka, Kansas; James W. Wilson, University of Georgia; Charles W. Trigg, San Diego, California; and the proposer.

A Smallest Partition

830. [March, 1972] *Proposed by Frank Dapkus, Seton Hall University.*

Find a right triangle with the smallest area that can be partitioned into two triangles with all integral sides.

Solution by J. W. Wilson, Athens, Georgia.

The hypotenuse of a right triangle is the diameter of its circumcircle. Hence the median to the hypotenuse is $\frac{1}{2}$ the length of the hypotenuse. Any right triangle with sides corresponding to a Pythagorean triple and the length of the hypotenuse an even number can thus be partitioned into two triangles with integral sides.

A Pythagorean triple is of the form

$$(a, b, c) = (2xy, x^2 - y^2, x^2 + y^2)$$

where x and y are integers and $x > y$. The first three Pythagorean triangles, in order of the area of the corresponding right triangles are (4, 3, 5), (6, 8, 10) and (12, 5, 13) corresponding to (x, y) of (2, 1), (3, 1) or (3, 2) respectively.

Now the (4, 3, 5) triangle cannot be partitioned into two triangles with integral sides. For, if the partition was accomplished by segment from one of the acute angles to a leg, then one of the two triangles in the partition would be a right triangle of smaller area and there is no right triangle with integral sides with an area less than that of the (4, 3, 5) triangle. If the partition was attempted by a segment from the right angle to the hypotenuse at a point n units from vertex of the 60° angle, $n = 1, 2, 3$ or 4, then the partitioning segment is of length k,

$$k = \sqrt{\left(4 - \frac{4n}{5}\right)^2 + \left(\frac{3n}{5}\right)^2}$$

and k is not an integer for $n = 1, 2, 3$ or 4.

The (6, 8, 10) triangle is the next largest and it can be partitioned into two triangles with integral sides by the median to the hypotenuse.

Also solved by Carl A. Argila, De La Salle College, Manila, Philippines; Leon Bankoff, Los Angeles, California; Merrill Barnebey, Holmen, Wisconsin; Michael Goldberg, Washington, D. C.; J. A. H. Hunter, Toronto, Canada; Ralph Jones, University of Massachusetts; Vaclav Konecny, Jarvis Christian College, Texas; K. R. S. Sastry, Makele, Ethiopia; E. P. Starke, Plainfield, New Jersey; Phil Tracy, Liverpool, New York; Charles W. Trigg, San Diego, California; Zalman Usiskin, University of Chicago; William Wernick, City College of New York; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

Comment on Problem 780

780. [November, 1970, and September, 1971] *Proposed by Simeon Reich, Israel Institute of Technology, Haifa, Israel.*

Let there be given a plane bounded closed convex set with interior points and with boundary of length p. If $p < \pi(2 + \sqrt{3})/3$, then one can rotate and translate this set in the plane so that in one position at least it will contain no lattice points.

Comment by the proposer.

Recall that a plane set K is said to be a covering set if for any position of K in the plane it contains at least one lattice point. Let $P = \inf\{p\colon p \text{ is the perimeter of } C$, a plane bounded closed convex covering set$\}$.

The bound $P \geqq 4$ obtained in [2] can be improved. Indeed, the fact that every eligible C must contain a unit square may be combined with the fact that a rectangle of dimensions a by b, $a \leqq b$, is a covering set if and only if $a \geqq 1$ and $b \geqq \sqrt{2}$, to yield $P \geqq 2 + 4\sqrt{1 - \sqrt{2}/2} \sim 4.16$.

Professor J. J. Schäffer (Carnegie-Mellon University) conjectures that the covering set which is smallest in area [1] is also smallest in perimeter. Its perimeter equals

$2+\sqrt{2}+\frac{1}{2}\log(3+2\sqrt{2})\sim 4.30$. Note that the isoperimetric inequality implies that $P \geqq 4\sqrt{\pi/3}\sim 4.09$.

I am indebted to Professors J. Hammer (University of Sydney) and J. J. Schäffer for their communications.

References

1. J. J. Schäffer, Smallest lattice-point covering convex set, Math. Ann., 129 (1955) 265–273.
2. H. G. Eggleston, Problems in Euclidean Space, Pergamon Press, New York, 1957.
3. Ivan Niven and H. S. Zuckerman, Lattice points covered by plane figures, Amer. Math. Monthly, 74 (1967) 354.

Comment on Problem 805

805. [September, 1971, and May, 1972] *Proposed by Charles W. Trigg, San Diego, California.*

Find the unique triangular number Δ_n which is a permutation of the ten digits and for which n has the form $abbbb$.

Comment by Cecil G. Phipps, Tennessee Technological University.

The statement of the problem is slightly incomplete or misleading. Furthermore, the solution, as printed, does not consider all possible cases.

A triangular number Δ_n having ten digits must have a value of n such that $4\times 10^4 < n < 14.2\times 10^4$. Under the restrictions given the number n must have either of two forms: (1) $abbbb$ or (2) $abbbbb$; whereas the statement of the problem suggests only three b's and the printed solution considers only the form with four b's.

Under the given conditions, a triangular number of form (1) must have $a \leqq 4$; and one of form (2) must have $abb < 142$. Hence under these conditions, in addition to the forms considered, one must also examine the forms $(1+5b)$ and $(1+5b+1)$ as multiples of 9. There are two such pairs, namely $(1,5)$ and $(1,7)$. But, since 15 and 17 are greater than 14, the corresponding triangular numbers have more than ten digits.

Comment on Problem 807

807. [September, 1971, and May, 1972] *Proposed by Norman Schaumberger Bronx Community College.*

Let (x_i), $i=1,2,3\ldots$ be an arbitrary sequence of positive real numbers, and set

$$\Delta_k = 1/k\sum_{i=1}^{k} x_i - \left(\prod_{i=1}^{k} x_i\right)^{1/k}.$$

If $n \geqq m$ prove that $n\Delta_n \geqq m\Delta_m$.

Comment by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.

The result here is known and is contained in a class of inequalities which are sometimes called Rado type inequalities (see D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, pp. 94, 98–102). Related to these inequalities are the ones analogous to

$$\left\{\frac{G_n(x)}{A_n(x)}\right\} \leqq \left\{\frac{G_{n-1}(x)}{A_{n-1}(x)}\right\}^{n-1}$$

which are sometimes called Popoviciu type inequalities. Here, G_n and A_n denote the geometric and arithmetic means of $x_1, x_2, \dots x_n$, respectively. A similar proof can also be given for the latter inequality. Just let

$$x_n = \lambda(x_1 + x_2 + \dots + x_{n-1})$$

giving

$$\frac{\lambda}{(1+\lambda)^n} \leqq \frac{(n-1)^{n-1}}{n^n}.$$

It follows easily that the r.h.s. is the maximum value of the l.h.s. which is taken on for $\lambda = 1/(n-1)$.

Comment on Q546

Q546. [May, 1972] If n is an integer greater than 2, prove that n is the sum of the nth powers of the roots of $x^n - kx - 1 = 0$.

[*Submitted by Erwin Just*]

Comment by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.

One can obtain further results in a similar fashion. If $T_1, T_2, \dots T_n$ denote the elementary symmetric functions of $x_1, x_2, \dots x_n$, i.e.,

$$P(x) = \Pi(x - x_i) = x^n - T_1 x^{n-1} + T_2 x^{n-2} - \dots + (-1)^n T_n$$

and if

$$S_k = \sum_{i=1}^{n} x_i^k$$

then the Newton formulae are given by

(A) $$S_k - T_1 S_{k-1} + T_2 S_{k-2} - \dots + (-1)^{k-1} T_{k-1} S_1 + (-1)^k k T_k = 0 \qquad (k \leqq n),$$

(B) $$S_k - T_1 S_{k-1} + T_2 S_{k-2} - \dots + (-1)^n T_n S_{k-n} = 0 \qquad (k > n).$$

If $P(x) \equiv x^n - ax - 1$, then $T_{n-1} = (-1)^n a$, $T_n = (-1)^{n-1}$ and $T_1 = T_2 = \dots T_{n-2} = 0$. It then follows that $S_m = 0$ for $m = rn + 1, rn + 2, \dots (r+1)n - r - 2$ $(1 \leqq r \leqq n-3)$. The nonvanishing power sums are given by

$$S_{n-1} = (n-1)a, \; S_n = n, \; S_{2n-2} = (n-1)a^2,$$

$$S_{2n-1} = (2n-1)a,\ S_{2n} = n,\ S_{3n-3} = (n-1)a^3,$$
$$S_{3n-2} = (3n-2)a^2,\ S_{3n-1} = (3n-1)a,\ S_{3n} = n, \text{ etc.}$$

ANSWERS

A557. A fourth power of an integer modulo 16 must be either 0 or 1. Thus, if a solution exists, then

$$x^4 + 2y^4 + 3z^4 - 4w^4 \equiv a_1 + 2a_2 + 3a_3 - 4a_4 \pmod{16}$$

in which each of the a_i is either 0 or 1.

Computation reveals that

$$a_1 + 2a_2 + 3a_3 - 4a_4 \equiv k \pmod{16}$$

in which the only possible values of k, modulo 16, are

$$0, \pm 1, \pm 2, \pm 3, \pm 4, 5 \text{ or } 6.$$

This proves that

$$x^4 + 2y^4 + 3z^4 - 4w^4 \equiv 7 \pmod{16}$$

has no solutions, and the desired conclusion is an immediate consequence.

A558. Let $R = (\cos a_1, \cos a_2, \ldots \cos a_n)$ denote the unit vector corresponding to the incoming ray. Here a_j denotes the angle R makes with the axis x_j. After reflection off any face (say the one normal to x_1), the new ray is given by

$$R = (-\cos a_1, \cos a_2 \ldots \cos a_n), \text{ etc.}$$

Consequently, after n reflections, the final ray is given by

$$R_n = (-\cos a_1, -\cos a_2 \ldots -\cos a_n) = -R.$$

A559. Squaring $a_{n+1}^2 - 10a_n + a_n^2 = 1$. Solving for a_n: $a_n = 5a_{n+1} - \sqrt{24a_{n+1}^2 + 1}$. Reducing n by one in the latter equation and adding it to the given equation we get $a_{n+1} = 10\, a_n - a_{n-1}$. Since $a_0 = 0$ and $a_1 = 1$ all the a_n's are integers.

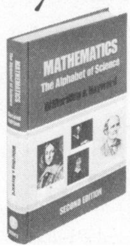
A560. No. Addition is not distributive over multiplication as $1 + 1 \cdot 1 \neq (1+1) \cdot (1+1)$ shows. In fact $(R^+, \cdot)$ is an abelian group and $(R^+, +)$ is a semigroup and the missing distributivity is the only missing property among ring properties. Here we do not require a ring to have an identity.

A561. No. If the mean of twin primes $a-1$ and $a+1$ is $a = b^2$ then $a - 1 = b^2 - 1 = (b+1)(b-1)$ which is a prime only $b = 2$ and $a = 4$. In fact no higher power can be a mean of twin primes. If it be c^{2n}, then $a - 1 = c^{2n} - 1$ which has a factor $c^n - 1$. If the power be c^{2n+1}, then $a + 1$ has a factor of $c + 1$. Each of these stated factors exceeds one.

(Quickies on page 43)

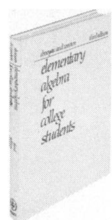
Math professors tell why they like:

Willerding & Hayward:



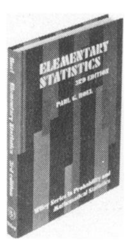
Mathematics: The Alphabet of Science, 2nd Edition ($9.95) Instructor's Manual.
"I like the fact that each chapter continues long enough to get into some of the interesting, advanced problems. This gives the better students a challenge. Also, if the whole class actually gets enthused over a section, it then becomes possible to pursue the matter further. I believe this last point is why we chose to adopt this text over the current one we are using." *—J. E. Koehler, Seattle Univ.*

Drooyan & Wooton:



Elementary Algebra for College Students, 3rd Edition ($8.95) Study Guide by Charles Carico.
"Very well written. Clear exposition. Chapter material presented in a logical manner. Sample problems presented in a manner which the student can understand."
—M. A. Chmielewski, Virginia State College

Hoel:



Elementary Statistics, 3rd Edition ($10.25)
Already in use at 150 colleges and universities!
"In the elementary statistics book, Hoel blends statistical theory with applications in an easy, straight-forward manner which is appealing to the concerned statistics student." *—R. M. Johnston, Midland Lutheran College*

For more information about these 3 successful math texts, contact your local Wiley representative, or write to Ben Bean, Dept. 2923, N. Y. office. Please include your course title, enrollment, and present text.

wiley **JOHN WILEY & SONS, Inc.**
605 Third Avenue, New York, N.Y. 10016
In Canada: 22 Worcester Road, Rexdale, Ontario

Prices subject to change without notice.

For clarity in counting . . .

HANDBOOK OF MATHEMATICAL TABLES AND FORMULAS, Fifth Edition
Richard S. Burington

1973, 500 pages, (009015-7), $5.50

This text has been designed to aid those in academic, professional, scientific, engineering, and business fields in which mathematical reasoning, processes, or computations are required. A serious effort has been made to retain information of a more traditional nature while incorporating those definitions, theorems, formulas, and tables needed for contemporary applications. Each subject treated is developed in a logical manner, to enable the user to interpret the information easily and properly.

BASIC MATHEMATICAL CONCEPTS, Second Edition
F. Lynwood Wren, California State University, Northridge

1973, 608 pages (tent.), (071907-1), $10.50

This text is designed for the study of those concepts of number, algebra, and geometry basic to intelligent teaching of mathematics in the elementary school. Beginning with the system of natural numbers, the author discusses its fundamental nature and then explains the inadequacies which call for extensions—to the domain of integers, the field of rational numbers, the field of real numbers, and finally the field of complex numbers. The remaining chapters present limited applications in the areas of geometry and algebra.

BASIC MATHEMATICAL CONCEPTS: A PRACTICE BOOK
F. Lynwood Wren, California State University, Northridge

1973, 150 pages (tent.), (071910-1), $3.50 (tent.)

Useful either in conjunction with or independent of the author's text, BASIC MATHEMATICAL CONCEPTS, Second Edition, this practice manual contains a number of approaches to the material differing sharply from the presentations in the text. The developmental pattern pursues two main purposes: to provide ample practice and review in basic operational procedures and to direct critical attention to the fundamental principles upon which these procedures are based.

FOUNDATIONS OF MATHEMATICS: With Applications in the Management and Social Sciences, Second Edition

Grace A Bush, Kent State University, and John E. Young, Southeast Missouri State College

1973, 466 pages (tent.), (009275-3), $11.50 (tent.)

An Instructor's Manual will be available.

Designed to introduce non-science students to the fundamentals of the real number system, this text includes applications of real numbers to such areas as probability, mathematics of finance, linear programming, matrices, calculus, and basic statistics. Little trigonometry is used.

. . . computing . . .

COLLEGE ALGEBRA

E. Richard Heineman, Texas Tech University

1973, 332 pages (tent.), (027936-5), $8.95
An Instructor's Manual will be available.

Designed for a freshman level algebra course, this text seeks to instill in students a realization that mathematics is a logical science and to develop their capability and understanding of those concepts which have traditionally constituted college algebra. The author achieves these goals by using modern terminology and the postulational approach to the properties of real numbers—together with clarity of presentation and carefully graded sets of diversified problems.

MODERN ALGEBRA AND TRIGONOMETRY, Second Edition

J. Vincent Robison, Emeritus, Oklahoma State University

1973, 431 pages, (053330-X), $9.50
An Instructor's Manual will be available.

This text develops and integrates traditional algebra and trigonometry through the use of concepts and techniques of set theory. The choice of topics, necessarily limited in a book of this length, has been influenced by the recommendations of both the Committee on the Undergraduate Program in Mathematics and the School Mathematics Study Group. Designed for students having no more than three semesters of high school algebra, the exposition is based on a judicious use of mathematical rigor and intuition, and a large number of worked examples illustrate concepts and provide methods of attack.

CONTEMPORARY TRIGONOMETRY

Howard E. Taylor, West Georgia College, and Thomas L. Wade, Florida State University

1973, 224 pages (tent.), (067640-2), $7.95

This text presents a concise, modern introduction to the field with emphasis on the trigonometric functions, their inverses, and their properties. The approach utilizes the basic concepts of the real number and rectangular coordinate systems, together with a small amount of set notation and a few elementary concepts of sets. Trigonometric functions and their inverses are viewed as non-empty sets of ordered pairs, no two of which have the same first entry. Historical notes are also provided.

The Bettmann Archive, Inc.